Equations and simple models

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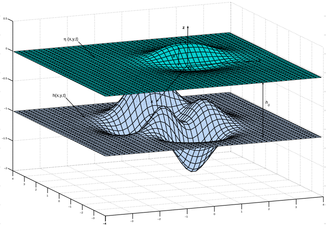
Hypothesis

Physical assumptions:

- Fluid is ideal, homogeneous & incompressible;
- Flow is irrotational;
- Free surface is a graph;
- Void above the free surface;
- Atmospheric pressure is constant.

Surface tension can also be included.



2D Problem over flat bottom

Hypothesis:

- Two-dimensional problem
- Perfect fluid with constant density ($\rho = 1$)
- Irrotational motion
- Flat horizontal bottom

Notations:

- d: mean depth
- g: acceleration due to gravity
- p: pressure divided by density
- x, y: horizontal and upward vertical coordinates
- u, v: horizontal and vertical velocities
- ϕ, ψ: velocity potential and stream function

Kinematic equations in the bulk

Incompressibility:

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = u_x + v_y = 0 \qquad \Rightarrow \qquad \exists\,\psi \;/\; \quad u = \psi_y, \quad v = -\psi_x.$$

Irrotationality:

$$v_x - u_y = 0 \qquad \Rightarrow \qquad \exists\,\phi \;/\; \quad u = \phi_x, \quad v = \phi_y.$$

Incompressibility+irrotationality:

$$\phi_x = \psi_y, \quad \phi_y = -\psi_x \qquad \Rightarrow \qquad \phi_{xx} + \phi_{yy} = \psi_{xx} + \psi_{yy} = 0.$$

Kinematic equations at the boundaries

Bottom impermeability:

$$v \;=\; 0 \qquad \text{at} \qquad y \;=\; -d.$$

Free surface impermeability at $y = \eta(x,t)$:

$$\frac{\mathrm{D}\,y}{\mathrm{D}t} \;=\; \frac{\mathrm{D}\,\eta}{\mathrm{D}t} \qquad \Rightarrow \qquad v \;=\; \eta_t \;+\; u\,\eta_x.$$

Temporal derivative following the motion:

$$\frac{\mathrm{D}}{\mathrm{D}t} \;=\; \frac{\partial}{\partial t} \;+\; u\,\frac{\partial}{\partial x} \;+\; v\,\frac{\partial}{\partial y},$$

with

$$u \;=\; \frac{\mathrm{D}\,x}{\mathrm{D}t}, \qquad v \;=\; \frac{\mathrm{D}\,y}{\mathrm{D}t}.$$

Euler–Lagrange equation

Dynamic equation at the free surface:

$$P \;=\; 0 \qquad \text{at} \qquad y \;=\; \eta(x,t).$$

Euler equation:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \;=\; -\,\frac{\boldsymbol{\nabla}P}{\rho} \;+\; \boldsymbol{g}.$$

With $p = P/\rho$ and $\boldsymbol{g} = (0, -g)$:

$$\begin{aligned} u_t \;+\; u\,u_x \;+\; v\,u_y \;&=\; -\,p_x, \\ v_t \;+\; u\,v_x \;+\; v\,v_y \;&=\; -\,p_y \;-\; g, \end{aligned}$$

For irrotational motions $\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = \boldsymbol{\nabla}(\frac{1}{2}|\boldsymbol{u}|^2)$, $\boldsymbol{u} = \boldsymbol{\nabla}\phi$:

$$\boldsymbol{\nabla}\left[\,\phi_t \;+\; \tfrac{1}{2}\,|\boldsymbol{\nabla}\phi|^2 \;+\; g\,y \;+\; p\,\right] \;=\; \boldsymbol{0}.$$

Euler–Lagrange equation (continued)

Cauchy–Lagrange equation:

$$\phi_t \,+\, \tfrac{1}{2}\,|\boldsymbol{\nabla}\phi|^2 \,+\, g\,y \,+\, p \,=\, C(t).$$

Change of potential:

$$\phi \,=\, \phi^* \,+\, \int C(t)\,\mathrm{d}t \qquad \Rightarrow \qquad \boldsymbol{\nabla}\phi \,=\, \boldsymbol{\nabla}\phi^*,$$

$$\Rightarrow \qquad \phi_t^* \,+\, \tfrac{1}{2}\,|\boldsymbol{\nabla}\phi^*|^2 \,+\, g\,y \,+\, p \,=\, 0.$$

Gauge condition:

$$C(t) \,=\, 0.$$

$C(t)$ can be taken equal to zero without loss of generality.
Does it mean that the $C(t)$ has completely disappeared?

Steady motions

Velocity fields independent of time:

$$\boldsymbol{u} \; = \; \boldsymbol{u}(\boldsymbol{x}) \qquad \text{or} \qquad \boldsymbol{u}_t \; = \; \boldsymbol{0}.$$

Thus $\boldsymbol{\nabla}\phi$ is independent of t, but ϕ ?

$$\phi(\boldsymbol{x},t) \; = \; \Phi(\boldsymbol{x}) \; + \; K(t) \qquad \Rightarrow \qquad \boldsymbol{\nabla}\phi \; = \; \boldsymbol{\nabla}\Phi \; = \; \boldsymbol{u}(\boldsymbol{x}),$$
$$\phi_t \; = \; \frac{\mathrm{d}\,K}{\mathrm{d}t} \; \neq \; 0.$$

Bernoulli equation

Cauchy–Lagrange equation:

$$\mathrm{d}K/\mathrm{d}t \;+\; \tfrac{1}{2}\,|\boldsymbol{\nabla}\Phi|^2 \;+\; g\,y \;+\; p \;=\; 0,$$

$$\Rightarrow \underbrace{\tfrac{1}{2}\,|\boldsymbol{\nabla}\Phi|^2 \;+\; g\,y \;+\; p}_{\text{independent of time}} \;=\; -\frac{\mathrm{d}\,K}{\mathrm{d}t} \;=\; \text{Constant} \;\equiv\; B,$$

$$\Rightarrow \qquad K \;=\; -\,B\,t \;+\; K_0.$$

Bernoulli equation:

$$\tfrac{1}{2}\,|\boldsymbol{\nabla}\Phi|^2 \;+\; g\,y \;+\; p \;=\; B.$$

Bernoulli equation with constant dissipation

Cauchy–Lagrange–Darcy equation:

$$\phi_t \;+\; \tfrac{1}{2}\,|\boldsymbol{\nabla}\phi|^2 \;+\; g\,y \;+\; p \;+\; \gamma\,\phi \;=\; C(t).$$

Gauge condition:

$$\phi \;=\; \phi^* \;+\; \int_{t_0}^{t} C(t')\,\mathrm{e}^{\gamma(t'-t)}\,\mathrm{d}t' \qquad \Rightarrow \qquad \boldsymbol{\nabla}\phi \;=\; \boldsymbol{\nabla}\phi^*,$$

$$\Rightarrow \qquad \phi_t^* \;+\; \tfrac{1}{2}\,|\boldsymbol{\nabla}\phi^*|^2 \;+\; g\,y \;+\; p \;+\; \gamma\,\phi^* \;=\; 0.$$

Bernoulli equation with variable dissipation

Change of potential:

$$\phi \;=\; \phi^* \;+\; \int_{t_0}^{t} C(t')\,\mathrm{e}^{\gamma(\boldsymbol{x})(t'-t)}\,\mathrm{d}t'$$

$$\Rightarrow \qquad \boldsymbol{\nabla}\phi \;=\; \boldsymbol{\nabla}\phi^* \;+\; \underbrace{(\boldsymbol{\nabla}\gamma)\int_{t_0}^{t} C(t')\,(t'-t)\,\mathrm{e}^{\gamma(\boldsymbol{x})(t'-t)}\,\mathrm{d}t'}_{\neq 0 \ \text{ if } \ \boldsymbol{\nabla}\gamma \neq \boldsymbol{0}},$$

$$\Rightarrow \qquad \phi_t^* \;+\; \tfrac{1}{2}\,|\boldsymbol{\nabla}\phi^*|^2 \;+\; g\,y \;+\; p \;+\; \gamma(\boldsymbol{x})\,\phi^* \;\neq\; 0.$$

Moral: Don't mess up with the Bernoulli constant!

Summary

Equations of motion:

$$
\begin{aligned}
\phi_{xx} + \phi_{yy} &= 0 && \text{for} && -d \leqslant y \leqslant \eta(x,t),\\
\phi_y &= 0 && \text{at} && y = -d,\\
\phi_y = \eta_t + \phi_x\,\eta_x & && \text{at} && y = \eta(x,t),\\
\phi_t + \tfrac{1}{2}\,|\boldsymbol{\nabla}\phi|^2 + g\,y + p &= 0 && \text{at} && y = \eta(x,t).
\end{aligned}
$$

Difficulties:

– Nonlinear terms in the surface boundary terms;

– Domain shape is unknown.

Approximation procedure

Trivial solution: $\phi = 0$, $\eta = 0$.

Perturbation scheme:

$$\phi \;=\; \epsilon\,\phi_1 \;+\; \epsilon^2\,\phi_2 \;+\; \cdots\,, \qquad \eta \;=\; \epsilon\,\eta_1 \;+\; \epsilon^2\,\eta_2 \;+\; \cdots\,,$$

ϵ: unspecified “small” parameter.

Taylor expansion around the rest level $y = 0$:

$$\phi(y=\eta) \;=\; \phi(y=0) \;+\; \eta\left[\frac{\partial\,\phi}{\partial y}\right]_{y=0} \;+\; \frac{\eta^2}{2}\left[\frac{\partial^2\,\phi}{\partial y^2}\right]_{y=0} \;+\; \cdots\,.$$

Perturbed equations

$$\epsilon\left[\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_1}{\partial y^2}\right]+\epsilon^2\left[\frac{\partial^2\phi_2}{\partial x^2}+\frac{\partial^2\phi_2}{\partial y^2}\right]+\cdots=0 \qquad -d\leqslant y\leqslant 0,$$

$$\epsilon\,\frac{\partial\phi_1}{\partial y}+\epsilon^2\,\frac{\partial\phi_2}{\partial y}+\cdots=0 \qquad y=-d,$$

$$\epsilon\left[\frac{\partial\eta_1}{\partial t}-\frac{\partial\phi_1}{\partial y}\right]+\epsilon^2\left[\frac{\partial\eta_2}{\partial t}+\frac{\partial\phi_1}{\partial x}\,\frac{\partial\eta_1}{\partial x}-\frac{\partial\phi_2}{\partial y}-\eta_1\,\frac{\partial^2\phi_1}{\partial y^2}\right]+\cdots=0 \qquad y=0,$$

$$\epsilon\left[\frac{\partial\phi_1}{\partial t}+g\eta_1\right]+\epsilon^2\left[\frac{\partial\phi_2}{\partial t}+\eta_1\,\frac{\partial^2\phi_1}{\partial y\,\partial t}+\frac{1}{2}\left(\frac{\partial\phi_1}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial\phi_1}{\partial y}\right)^2+g\eta_2\right]+\cdots=0 \qquad y=0.$$

First-order approximation: Linear theory

Linearised equations:

$$
\begin{aligned}
\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} &= 0 \qquad -d \leqslant y \leqslant 0, \\
\frac{\partial \phi_1}{\partial y} &= 0 \qquad y = -d, \\
\frac{\partial \eta_1}{\partial t} - \frac{\partial \phi_1}{\partial y} &= 0 \qquad y = 0, \\
\frac{\partial \phi_1}{\partial t} + g\,\eta_1 &= 0 \qquad y = 0.
\end{aligned}
$$

Elimination of η:

$$
\eta_1 = -\frac{1}{g}\left.\frac{\partial \phi_1}{\partial t}\right|_{y=0} \qquad \Rightarrow
$$

$$
g\,\frac{\partial \phi_1}{\partial y} + \frac{\partial^2 \phi_1}{\partial t^2} = 0 \qquad \text{at} \quad y = 0.
$$

Resolution of the linear equations

Separation of dependent variables:

$$\phi_1(x,y,t) \;=\; X(x)\,Y(y)\,T(t) \qquad \Rightarrow \qquad \eta_1 \;=\; -\,\frac{Y(0)}{g}\,X(x)\,T'(t).$$

Laplace equation:

$$\begin{aligned} X''(x)\,Y(y)\,T(t) \;+\; X(x)\,Y''(y)\,T(t) \;&=\; 0 \\ \Rightarrow \qquad \frac{X''(x)}{X(x)} \;=\; -\,\frac{Y''(y)}{Y(y)} \;&=\; \text{constant} \;\equiv\; C_1, \end{aligned}$$

Bottom impermeability:

$$X(x)\,Y'(-d)\,T(t) \;=\; 0 \qquad \Rightarrow \qquad Y'(-d) \;=\; 0.$$

Surface impermeability:

$$\begin{aligned} g\,X(x)\,Y'(0)\,T(t) \;+\; X(x)\,Y(0)\,T''(t) \;&=\; 0 \\ \Rightarrow \qquad \frac{T''(t)}{T(t)} \;=\; -\,g\,\frac{Y'(0)}{Y(0)} \;&\equiv\; C_2. \end{aligned}$$

First case: $C_1 = -k^2 < 0$, $C_2 = -\omega^2 < 0$

Solutions of the eigen problem:

$$X(x) \;=\; \left\{ \begin{array}{c} \cos kx \\ \sin kx \end{array} \right\}, \qquad Y(y) \;\propto\; \cosh k(y+d), \qquad T(t) \;=\; \left\{ \begin{array}{c} \cos \omega t \\ \sin \omega t \end{array} \right\},$$

$\{\ \}$: arbitrary linear combination.

Dispersion relation:

$$\omega^2 \;=\; gk \tanh kd,$$

only one solution $k = k_0 > 0$ for fixed ω.

Second case: $C_1 = k^2 > 0$, $C_2 = -\omega^2 < 0$

Solutions of the eigen problem:

$$X(x) = \left\{ \begin{matrix} \mathrm{e}^{kx} \\ \mathrm{e}^{-kx} \end{matrix} \right\}, \quad Y(y) \propto \cos k(y+d), \quad T(t) = \left\{ \begin{matrix} \cos \omega t \\ \sin \omega t \end{matrix} \right\},$$

$\{\ \}$: arbitrary linear combination.

Dispersion relation:

$$\omega^2 = -gk \tan kd,$$

infinite number of reel solutions $k = k_n > 0$ for fixed ω.

General linear solution

Superposition:

$$
\begin{aligned}
\phi_1 = & \int_0^\infty \left\{ \begin{array}{c} \cos k_0 x \\ \sin k_0 x \end{array} \right\} \cosh k_0(y+d) \left\{ \begin{array}{c} \cos \omega t \\ \sin \omega t \end{array} \right\} \mathrm{d}\omega \\
& + \sum_{n=1}^{\infty} \int_0^\infty \left\{ \begin{array}{c} \mathrm{e}^{k_n x} \\ \mathrm{e}^{-k_n x} \end{array} \right\} \cos k_n(y+d) \left\{ \begin{array}{c} \cos \omega t \\ \sin \omega t \end{array} \right\} \mathrm{d}\omega,
\end{aligned}
$$

with

$$
\omega^2 = g k_0 \tanh k_0 d, \qquad \omega^2 = -g k_n \tan k_n d \quad (n = 1, 2, \cdots).
$$

Blue: wave modes.
Red: evanescent modes.

Linear standing wave

Clapotis:

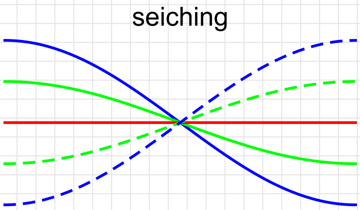
$$
\begin{aligned}
\phi_1 &= A\ \cos k_0 x\ \cosh k_0(y+d)\ \sin\omega t, \\
\eta_1 &= -\,\omega\, A\, g^{-1}\ \cosh k_0 d\ \cos k_0 x\ \cos\omega t, \\
u_1 &= \phi_x\ =\ -k_0 A\ \sin k_0 x\ \cosh k_0(y+d)\ \sin\omega t, \\
v_1 &= \phi_y\ =\ k_0 A\ \cos k_0 x\ \sinh k_0(y+d)\ \sin\omega t.
\end{aligned}
$$

Stream lines ($\mathrm{d}x/u = \mathrm{d}y/v$):

$$
\frac{\mathrm{d}x}{\tan k_0 x} = \frac{-\,\mathrm{d}y}{\tanh k_0(y+d)} \quad \Rightarrow \quad \sin k_0 x\ \sinh k_0(y+d) = \text{Cst.}
$$

The streamlines are independent of time.

Examples



Remarks on nonlinear standing waves

Fourth-order theory shows that:

– Anti-nodes remain;

– There are no nodes;

– Free surface is never flat.

Traveling waves

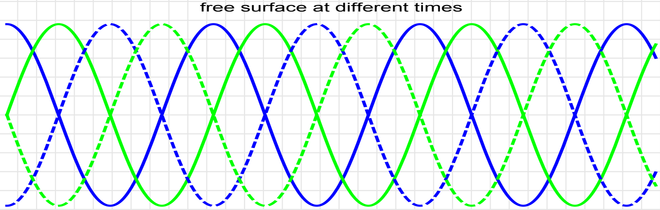
Clapotis + clapotis:

$$\begin{aligned}\phi_1 \;&=\; A\ \sin k_0 x\ \cosh k_0(y+d)\ \cos\omega t \\ &\quad -\ A\ \cos k_0 x\ \cosh k_0(y+d)\ \sin\omega t \\ &=\; A\ \cosh k_0(y+d)\ \sin(k_0 x-\omega t),\end{aligned}$$

with $\omega^2 = g k_0 \tanh k_0 d$.

Free surface:

$$\eta_1 \;=\; (\omega A/g)\ \cosh k_0 d\ \cos(k_0 x-\omega t).$$



Physical parameters

First-order approximation:

$$\phi \approx \epsilon\,\phi_1 = \epsilon\,A\,\cosh k_0(y+d)\,\sin(k_0x-\omega t),$$
$$\eta \approx \epsilon\,\eta_1 = \epsilon\,(\omega A/g)\,\cosh k_0 d\,\cos(k_0x-\omega t).$$

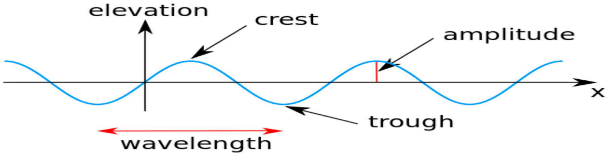
Wavelength $2\pi/k_0$, Period $2\pi/\omega$, Phase velocity $c=\omega/k_0$

.

$$\frac{c^2}{c_0^2} = \frac{\tanh k_0 d}{k_0 d} \leqslant 1, \qquad c_0 \equiv \sqrt{gd}.$$

Amplitude $a=\max(\eta)$:

$$a = \epsilon\,(\omega A/g)\,\cosh k_0 d.$$



Limiting cases

Deep water $d \to \infty$:

$$\phi \;\to\; (ga/\omega)\, \exp(k_0 y)\, \sin(k_0 x - \omega t + \delta), \qquad \omega^2 \;\to\; g\, k_0.$$

Shallow water $k_0 \to 0$:

$$\phi \;=\; (ga/c_0 k_0)\, \sin(k_0 x - \omega t + \delta) \;+\; O(k_0^2), \qquad \omega^2 \;\to\; c_0^2\, k_0^2.$$

In shallow water, the linear theory describes waves of zero amplitude (uniform current) and thus cannot describe solitary waves.

Particle trajectories

Equations of trajectories:

$$\frac{\mathrm{D}\,x}{\mathrm{D}t} \;=\; \frac{\partial\,\phi}{\partial x}, \qquad \frac{\mathrm{D}\,y}{\mathrm{D}t} \;=\; \frac{\partial\,\phi}{\partial y}.$$

Particles closed to their mean position $x = \alpha$, $y = \beta$:

$$x \;\approx\; \alpha \;+\; \epsilon\, x_1(\alpha,\beta,t), \qquad y \;\approx\; \beta \;+\; \epsilon\, y_1(\alpha,\beta,t).$$

Linearised equations:

$$\epsilon\,\frac{\partial\,x_1}{\partial t} \;\approx\; \epsilon\,\frac{\partial\,\phi_1}{\partial x}\bigg|_{\substack{x=\alpha\\ y=\beta}}, \qquad \epsilon\,\frac{\partial\,y_1}{\partial t} \;\approx\; \epsilon\,\frac{\partial\,\phi_1}{\partial y}\bigg|_{\substack{x=\alpha\\ y=\beta}}.$$

Trajectories under travelling waves

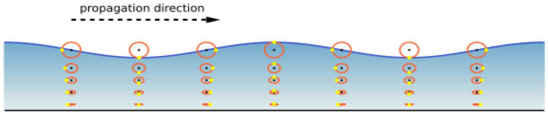
Solution of the linearised equations:

$$x = \alpha - a \sin(k_0\alpha - \omega t + \delta) \frac{\cosh k_0(\beta + d)}{\sinh k_0 d},$$
$$y = \beta + a \cos(k_0\alpha - \omega t + \delta) \frac{\sinh k_0(\beta + d)}{\sinh k_0 d},$$

thence

$$\frac{(x-\alpha)^2}{\cosh^2 k_0(\beta+d)} + \frac{(y-\beta)^2}{\sinh^2 k_0(\beta+d)} = \frac{a^2}{\sinh^2 k_0 d}.$$

Trajectories are closed ellipses. (Circles in deep water.)



Energy of pure swell

Swell linear approximation:

$$\phi \;=\; A\,\cos(kx-\omega t)\,\cosh k(y+d), \qquad \eta \;=\; a\,\sin(kx-\omega t),$$
$$\omega^2 \;=\; gk\tanh kd, \qquad A \;=\; -g\,a\,/\,\omega\cosh kd.$$

Kinetic energy:

$$\begin{aligned}
E_{\mathsf{K}} \;&\equiv\; \int_{-\frac{\pi}{k}}^{\frac{\pi}{k}}\int_{-d}^{\eta}\frac{u^2+v^2}{2}\,\mathrm{d}y\,\mathrm{d}x \;\approx\; \int_{-\frac{\pi}{k}}^{\frac{\pi}{k}}\int_{-d}^{0}\frac{u^2+v^2}{2}\,\mathrm{d}y\,\mathrm{d}x \\
&=\; \int_{-\frac{\pi}{k}}^{\frac{\pi}{k}}\int_{-d}^{0}\frac{A^2\,k^2}{2}\,\Big[\sin^2(kx-\omega t)\cosh^2 k(y+d) \\
&\qquad +\cos^2(kx-\omega t)\sinh^2 k(y+d)\Big]\,\mathrm{d}y\,\mathrm{d}x \\
&=\; \tfrac{1}{4}\,\pi\,A^2\sinh 2kd.
\end{aligned}$$

Potential energy:

$$E_\mathsf{P} \;\equiv\; \int_{-\frac{\pi}{k}}^{\frac{\pi}{k}} \int_{-d}^{\eta} g\,y\,\mathrm{d}y\,\mathrm{d}x \;-\; \int_{-\frac{\pi}{k}}^{\frac{\pi}{k}} \int_{-d}^{0} g\,y\,\mathrm{d}y\,\mathrm{d}x \;=\; \int_{-\frac{\pi}{k}}^{\frac{\pi}{k}} \int_{0}^{\eta} g\,y\,\mathrm{d}y\,\mathrm{d}x$$

$$=\; \int_{-\frac{\pi}{k}}^{\frac{\pi}{k}} \tfrac{1}{2}\,g\,\eta^2\;\mathrm{d}x \;\approx\; \frac{\pi\,A^2\,\omega^2\cosh^2 kd}{2\,g\,k}.$$

Remark:

$$\frac{E_\mathsf{K}}{E_\mathsf{P}} \;=\; \frac{g\,k\tanh kd}{\omega^2} \;=\; 1,$$

Energy flux of pure swell

Flux through a vertical:

$$q_{\mathsf{E}} \;\equiv\; \int_{-d}^{\eta} p\,u\,\mathrm{d}y \;\approx\; \int_{-d}^{0} -\left[\frac{\partial\,\phi}{\partial t} + gy\right]\frac{\partial\,\phi}{\partial x}\,\mathrm{d}y.$$

Mean flux:

$$Q_{\mathsf{E}} \;\equiv\; \frac{\omega}{2\,\pi}\int_{t}^{t+\frac{2\pi}{\omega}} q_{\mathsf{E}}\,\mathrm{d}t \;\approx\; \frac{k\,d\,\omega\,A^2}{4}\left[1 \,+\, \frac{\sinh 2kd}{2\,k\,d}\right].$$

Group velocity

Time of energy transport over a wavelength $L = 2\pi/k$:

$$T_{\mathsf{E}} \equiv \frac{E_{\mathsf{K}} + E_{\mathsf{P}}}{Q_{\mathsf{E}}}.$$

Group velocity:

$$c_g \equiv \frac{L}{T_{\mathsf{E}}} = \frac{Q_{\mathsf{E}}\, L}{E_{\mathsf{K}} + E_{\mathsf{P}}} \approx \frac{c}{2}\left[1 + \frac{2kd}{\sinh 2kd}\right] = \frac{\partial\,\omega}{\partial k}.$$

– Deep water: $c_g = c/2$.
– Shallow water: $c_g = c$.

Higher-order theories

At order ϵ^2:

– Double frequency appears (first harmonic).

At order ϵ^3:

– Triple frequency appears (second harmonic).
– Phase velocity depends on the amplitude.

With 'slow' amplitude modulation:

– Nonlinear Schrödinger equation.
– Dysthe's equation.