Shallow water theories

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Shallow water

Long wave assumption:

– Typical wavelength L is large compared to the mean depth d, i.e., $d/L \ll 1$.
– Typical period T is large too because phase velocities are closed to the critical speed $c_0 = \sqrt{gd}$, i.e, $L/T \approx c_0$.

Distorsion:

$$\xi = \epsilon\, x, \quad \tau = \epsilon\, t \qquad \Rightarrow \qquad \frac{\partial}{\partial x} = \epsilon\, \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial t} = \epsilon\, \frac{\partial}{\partial \tau}.$$

Shallow water equations

$$
\begin{aligned}
\epsilon^2 \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial y^2} &= 0 \qquad -d \leqslant y \leqslant \eta, \\
\frac{\partial \phi}{\partial y} &= 0 \qquad y = -d, \\
g\,\eta + \epsilon \frac{\partial \phi}{\partial \tau} + \frac{\epsilon^2}{2}\left(\frac{\partial \phi}{\partial \xi}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial y}\right)^2 &= 0 \qquad y = \eta, \\
\frac{\partial \phi}{\partial y} - \epsilon \frac{\partial \eta}{\partial \tau} - \epsilon^2 \frac{\partial \phi}{\partial \xi}\frac{\partial \eta}{\partial \xi} &= 0 \qquad y = \eta.
\end{aligned}
$$

Small parameter expansion:

$$
\phi = \epsilon\,\phi_1 + \epsilon^3\,\phi_3 + \cdots, \qquad \eta = \epsilon^2\,\eta_2 + \epsilon^4\,\eta_4 + \cdots.
$$

Resolution at orders 1 and 2

Laplace equation and bottom impermeability at order ϵ:

$$\frac{\partial^2 \phi_1}{\partial y^2} = 0, \qquad \left.\frac{\partial \phi_1}{\partial y}\right|_{y=-d} = 0,$$

$$\Rightarrow \qquad \phi_1 = \Phi_1(\xi, \tau).$$

Surface isobarity at order ϵ^2:

$$g\,\eta_2 + \left.\frac{\partial \phi_1}{\partial \tau}\right|_{y=0} = 0 \qquad \Rightarrow \qquad \eta_2 = -\frac{1}{g}\,\frac{\partial \Phi_1}{\partial \tau}.$$

Resolution at order 3

Laplace equation and bottom impermeability at order ϵ^3:

$$\frac{\partial^2 \phi_1}{\partial \xi^2} + \frac{\partial^2 \phi_3}{\partial y^2} = 0, \qquad \left.\frac{\partial \phi_3}{\partial y}\right|_{y=-d} = 0,$$

$$\Rightarrow \qquad \phi_3 = \Phi_3(\xi, \tau) - \frac{(y+d)^2}{2} \frac{\partial^2 \Phi_1}{\partial \xi^2}.$$

Surface impermeability at order ϵ^3:

$$\left.\frac{\partial \phi_3}{\partial y}\right|_{y=0} = \frac{\partial \eta_2}{\partial \tau} \qquad \Rightarrow \qquad \frac{\partial^2 \Phi_1}{\partial \tau^2} - g\, d\, \frac{\partial^2 \Phi_1}{\partial \xi^2} = 0,$$

$$\Rightarrow \qquad \Phi_1 = \Phi_1^-(\xi - c\tau) + \Phi_1^+(\xi + c\tau), \qquad c \equiv \sqrt{gd}.$$

Special case:

$$\Phi_1 = \Phi_1(\xi - c\tau).$$

Resolution at order 4 and 5

Surface isobarity at order ϵ^4:

$$g\,\eta_4 \;+\; \left[\,\frac{\partial\,\phi_3}{\partial\tau} + \frac{1}{2}\left(\frac{\partial\,\phi_1}{\partial\xi}\right)^2\,\right]_{y=0} \;=\; 0$$

$$\Rightarrow \qquad \eta_4 \;=\; -\,\frac{1}{g}\,\frac{\partial\,\Phi_3}{\partial\tau} \;+\; \frac{d^2}{2g}\,\frac{\partial^3\,\Phi_1}{\partial^2\xi\,\partial\tau} \;-\; \frac{1}{2g}\left(\frac{\partial\,\Phi_1}{\partial\xi}\right)^2.$$

Laplace equation and bottom impermeability at order ϵ^5:

$$\frac{\partial^2\,\phi_3}{\partial\xi^2} + \frac{\partial^2\,\phi_5}{\partial y^2} \;=\; 0, \qquad \left.\frac{\partial\,\phi_5}{\partial y}\right|_{y=-d} \;=\; 0,$$

$$\Rightarrow \qquad \phi_5 \;=\; \Phi_5(\xi,\tau) \;-\; \frac{(y+d)^2}{2}\,\frac{\partial^2\,\Phi_3}{\partial\xi^2} \;+\; \frac{(y+d)^4}{4!}\,\frac{\partial^4\,\Phi_1}{\partial\xi^4}.$$

Surface impermeability at order ϵ^5:

$$\left.\frac{\partial\,\phi_5}{\partial y}\right|_{y=0} \;+\; \eta_2\,\left.\frac{\partial\,\phi_3}{\partial y}\right|_{y=0} \;=\; \frac{\partial\,\eta_4}{\partial \tau} \;+\; \frac{\partial\,\eta_2}{\partial \xi}\,\left.\frac{\partial\,\phi_3}{\partial \xi}\right|_{y=0}.$$

Special case $\Phi_n = \Phi_n(\theta)$ with $\theta \equiv \xi - c\tau$:

$$\frac{c\,d^2}{6}\,\frac{\mathrm{d}^4\,\Phi_1}{\mathrm{d}\theta^4} \;+\; \frac{3}{2}\,\frac{\mathrm{d}\,\Phi_1}{\mathrm{d}\theta}\,\frac{\mathrm{d}^2\,\Phi_1}{\mathrm{d}\theta^2} \;=\; 0,$$

or with $\eta_2 = (c/g)\,\mathrm{d}\Phi_1/\mathrm{d}\theta$:

$$\frac{c\,d^2}{6}\,\frac{\mathrm{d}^3\,\eta_2}{\mathrm{d}\theta^3} \;+\; \frac{3\,g}{2\,c}\,\eta_2\,\frac{\mathrm{d}\,\eta_2}{\mathrm{d}\theta} \;=\; 0.$$

General resolution for travelling waves

First integration:

$$\frac{c\,d^2}{6}\,\frac{\mathrm{d}^2\,\eta_2}{\mathrm{d}\theta^2} \;+\; \frac{3\,g}{4\,c}\,\eta_2^2 \;=\; \text{constant} \;\equiv\; K_1.$$

Second integration (after multiplication by η_2'):

$$\frac{c\,d^2}{12}\left(\frac{\mathrm{d}\,\eta_2}{\mathrm{d}\theta}\right)^2 \;+\; \frac{g}{4\,c}\,\eta_2^3 \;=\; K_1\,\eta \;+\; K_2.$$

General solution via Jacobian elliptic functions.

Solitary wave

Ansatz: $\eta_2 = \alpha_0 + \alpha_1 \operatorname{sech}^2(\alpha_2\theta)$.

After substitution into the equation:

$$\alpha_2 = \frac{\sqrt{3\,g\,\alpha_1}}{2\,d\,c}, \quad \alpha_0 = -\frac{\alpha_1}{3}, \quad K_1 = \frac{g\,\alpha_1^2}{12\,c}, \quad K_2 = \frac{g\,\alpha_1^3}{54\,c}.$$

Problem: $\alpha_0 = K_1 = K_2 = 0$ only if $\alpha_1 = 0$, that is for zero amplitude!?

Remedy: renormalisation. Observations:

- d is **not** the mean depth!
- c is **not** the phase velocity!

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Approximate solution

Free surface:

$$\eta \;\approx\; \epsilon^2\,\eta_2 \;=\; \epsilon^2\,\alpha_0 \;+\; \epsilon^2\,\alpha_1\,\operatorname{sech}^2(\alpha_2\theta).$$

Horizontal velocity:

$$u \;\approx\; \epsilon^2\,\frac{\mathrm{d}\,\Phi_1}{\mathrm{d}\theta} \;=\; \epsilon^2\,\frac{g\,\eta_2}{c} \;\approx\; \frac{g\,\eta}{c}.$$

Physical parameters

Total local water depth: $h \equiv d + \eta$.

– Mean physical (or real) depth d^* for a solitary wave such that: $h(\infty) = d^*$. Thence:

$$d^* = d + \epsilon^2 \alpha_0 \qquad \Rightarrow \qquad d = d^* + \epsilon^2 \alpha_1 / 3.$$

– Free surface elevation from rest:

$$\eta^* \equiv h(\theta) - h(\infty) \approx \epsilon^2 \alpha_1 \operatorname{sech}^2(\alpha_2 \theta).$$

– Wave amplitude $a \equiv \eta^*(0)$, thence $a = \epsilon^2 \alpha_1$.

Apparent phase velocity:

$$c \;=\; \sqrt{g\,d} \;=\; \sqrt{g\,(d^* + \epsilon^2\alpha_1/3)} \;\approx\; \sqrt{g\,d^*}\left(1 \,+\, \frac{\epsilon^2\,\alpha_1}{6\,d^*}\right).$$

'Fixed' reference frame: $u^* = u - u(\infty)$.

Galilean transformation:

$$\begin{aligned}\theta^* \;&=\; \theta \,+\, u(\infty)\,\tau \;=\; \xi \,-\, c\,\tau \,+\, \epsilon^2\,(g/c)\,\alpha_0\,\tau \\ &=\; \xi \,-\, \left(c \,+\, \epsilon^2\,\frac{g}{3\,c}\,\alpha_1\right)\tau.\end{aligned}$$

Phase velocity in the 'fixed' frame of reference:

$$\begin{aligned}c^* \;\equiv\; c \,+\, \epsilon^2\,\frac{g}{3\,c}\,\alpha_1 \;&\approx\; \sqrt{g\,d^*}\left(1 \,+\, \frac{\epsilon^2\,\alpha_1}{2\,d^*}\right) \\ &=\; \sqrt{g\,d^*}\left(1 \,+\, \frac{a}{2\,d^*}\right).\end{aligned}$$

Solitary wave charactistics

Froude number:

$$\mathsf{Fr} \equiv \frac{c^*}{\sqrt{g\,d^*}} \approx 1 + \frac{a}{2\,d^*} \geqslant 1.$$

Solitary wave speed is super-critical.

Summary

Approximations break symmetries, in general.

Parameters may not be what they seem to be.

(Functions and operators may have the same 'problem'.)

A posteriori check of parameters definition.

(Parameters must be well defined!)

Renormalise, if needed.

(Removal of secular terms. We must know/understand the Physics!)

Other techniques: Multiple scales, Lie group transformations, etc.

Multiple scale variant

Distorsion for almost unidirectional waves:

$$\xi \;=\; \epsilon\,(\,x\,-\,c_0\,t\,), \qquad \tau \;=\; \epsilon^3\,t, \qquad c_0 \;=\; \sqrt{g\,d}\,,$$
$$\Rightarrow \qquad \frac{\partial}{\partial x} \;=\; \epsilon\,\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial t} \;=\; \epsilon^3\,\frac{\partial}{\partial \tau} \;-\; \epsilon\,c_0\,\frac{\partial}{\partial \xi}.$$

Small parameter expansion:

$$\phi \;=\; \epsilon\,\phi_1 \;+\; \epsilon^3\,\phi_3 \;+\; \cdots\,, \qquad \eta \;=\; \epsilon^2\,\eta_2 \;+\; \epsilon^4\,\eta_4 \;+\; \cdots\,.$$

Solution of Laplace and bottom equations:

$$\phi_1 \;=\; \Phi_1(\xi,\tau), \qquad \phi_3 \;=\; \Phi_3(\xi,\tau) \;-\; \frac{(y+d)^2}{2}\,\frac{\partial^2\,\Phi_1}{\partial \xi^2},$$
$$\phi_5 \;=\; \Phi_5(\xi,\tau) \;-\; \frac{(y+d)^2}{2}\,\frac{\partial^2\,\Phi_3}{\partial \xi^2} \;+\; \frac{(y+d)^4}{4!}\,\frac{\partial^4\,\Phi_1}{\partial \xi^4}.$$

Surface isobarity:

$$\eta_2 = \frac{c_0}{g}\,\frac{\partial\,\Phi_1}{\partial\xi},$$

$$\eta_4 = \frac{c_0}{g}\,\frac{\partial\,\Phi_3}{\partial\xi} - \frac{c_0\,d^2}{2\,g}\,\frac{\partial^3\,\Phi_1}{\partial\xi^3} - \frac{1}{g}\,\frac{\partial\,\Phi_1}{\partial\tau} - \frac{1}{2\,g}\left(\frac{\partial\,\Phi_1}{\partial\xi}\right)^2.$$

Surface impermeability at order ϵ^5:

$$\frac{\partial\,\eta_2}{\partial\tau} + \frac{c_0\,d^2}{6}\,\frac{\partial^3\,\eta_2}{\partial\xi^3} + \frac{3\,g}{2\,c_0}\,\eta_2\,\frac{\partial\,\eta_2}{\partial\xi} = 0.$$

Korteweg & de Vries equation

Approximation:

$$\eta \;\approx\; \epsilon^2\,\eta_2, \qquad \frac{\partial}{\partial\xi} \;=\; \frac{1}{\epsilon}\,\frac{\partial}{\partial x},$$

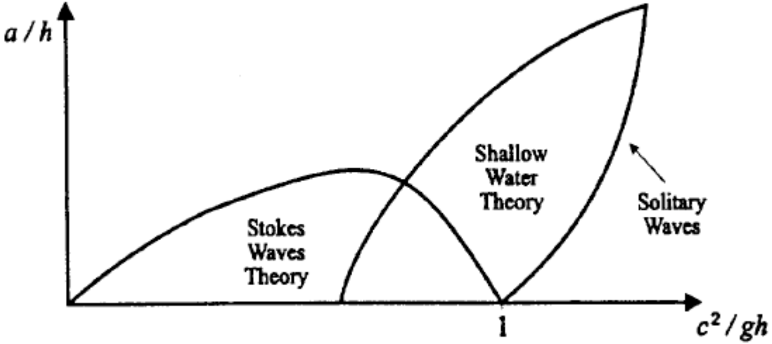
$$\frac{\partial}{\partial\tau} \;=\; \frac{1}{\epsilon^3}\,\frac{\partial}{\partial t} \;+\; \frac{c_0}{\epsilon^2}\,\frac{\partial}{\partial\xi} \;=\; \frac{1}{\epsilon^3}\,\frac{\partial}{\partial t} \;+\; \frac{c_0}{\epsilon^3}\,\frac{\partial}{\partial x},$$

KdV:

$$\frac{\partial\,\eta}{\partial t} \;+\; c_0\,\frac{\partial\,\eta}{\partial x} \;+\; \frac{3\,g}{2\,c_0}\,\eta\,\frac{\partial\,\eta}{\partial x} \;+\; \frac{c_0\,d^2}{6}\,\frac{\partial^3\,\eta}{\partial x^3} \;=\; 0.$$

Validity of shallow water theories

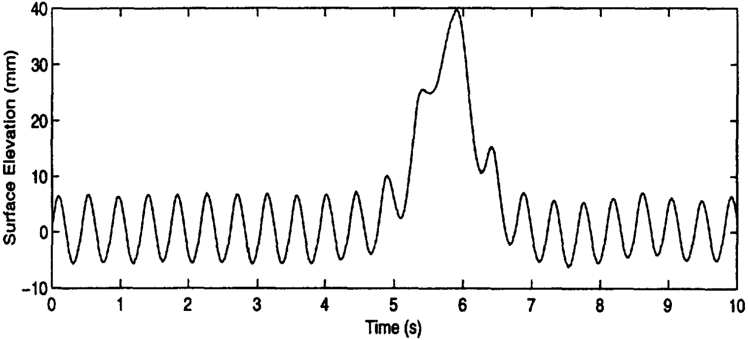
Littman's diagram:



– Stokes expansion: finite radius of convergence.
– Shallow water expansion: divergent series.

Sinusoidal wave over a solitary wave

Experimental measurements:



Hybrid expansion (Hint of the method)

Stokes series:

$$\phi \;=\; \sum_{m=1}^{\infty} \sum_{n=-m}^{m} \epsilon^{m}\, A_{m,n}(y)\, \mathrm{e}^{n\mathrm{i}(kx-\omega t)}.$$

With ‘slow’ variables $\xi = \epsilon x, \;\; \tau = \epsilon t$:

$$\phi \;=\; \sum_{m=1}^{\infty} \sum_{n=-m}^{m} \epsilon^{m}\, A_{m,n}(y)\, \mathrm{e}^{n\mathrm{i}(k\xi-\omega\tau)/\epsilon}.$$

Nonlinear WKB expansion:

$$\phi \;=\; \sum_{m=1}^{\infty} \sum_{n=-m}^{m} \epsilon^{m}\, A_{m,n}(\xi, y, \tau)\, \mathrm{e}^{n\mathrm{i}S(\xi,\tau)/\epsilon},$$

$$S \;=\; \sum_{m=0}^{\infty} \epsilon^{m}\, S_{m}(\xi, \tau).$$

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Amplitude vs Phase

Phase expansion, e.g.:

$$\exp(n\,\mathrm{i}\,\epsilon\,S_2) \;=\; 1 \;+\; n\,\mathrm{i}\,\epsilon\,S_2 \;+\; O(\epsilon^2).$$

Phase terms can be rewritten as amplitude terms.

Similarly, amplitude terms can be rewritten as phase terms.

Lagrange solution of Laplace equation

Laplace + bottom impermeability:

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{for} \quad y \geqslant 0, \qquad \phi_y = 0 \quad \text{at} \quad y = -d.$$

Lagrange (1791) expansion:

$$\phi(x,y,t) = \sum_{n=0}^{\infty} (y+d)^n \, f_n(x,t) \qquad \Rightarrow$$

$$\sum_{n=0}^{\infty} (y+d)^n \, \frac{\partial^2 f_n}{\partial x^2} + \sum_{n=2}^{\infty} n\,(n-1)\,(y+d)^{n-2} f_n = 0$$

$$\sum_{n=0}^{\infty} (y+d)^n \left[\frac{\partial^2 f_n}{\partial x^2} + (n+2)\,(n+1)\,f_{n+2} \right] = 0.$$

with bottom impermeability:

$$\begin{aligned} f_0 &= \phi(x, y=-d, t) \equiv \Phi(x,t), \qquad f_1 = 0, \\ \phi(x,y,t) &= \sum_{n=0}^{\infty} (-1)^n \, \frac{(y+d)^{2n}}{(2n)!} \, \frac{\partial^{2n}\,\Phi(x,t)}{\partial x^{2n}} \\ &= \cos[\,(y+d)\,\partial_x\,]\,\Phi(x,t). \end{aligned}$$

If the potential is known at the bottom, it is known everywhere.

If $\Phi \propto \sin(kx - \omega t)$:

$$\begin{aligned} \phi(x,y,t) &= \cos[\,(y+d)\,\partial_x\,]\,\sin(kx-\omega t) \\ &= \cosh k(y+d)\,\sin(kx-\omega t) \end{aligned}$$

If $\Phi \propto \tanh(kx - \omega t) \;\Rightarrow$ complicated (convergent?) series.

General solution of Laplace equation

Laplace equation rewritten:

$$\phi_{xx} \; - \; \mathrm{i}^2\,\phi_{yy} \; = \; 0, \qquad \mathrm{i}^2 \; = \; -1,$$

$$\Rightarrow \qquad \phi \; = \; F(x+\mathrm{i}(y+d)) \; + \; G(x-\mathrm{i}(y+d)).$$

Bottom impermeability:

$$0 \; = \; \phi_y(y=-d) \; = \; \mathrm{i}\,F'(x) \; - \; \mathrm{i}\,G'(x)$$

$$\Rightarrow \qquad G \; = \; F \; \equiv \; \tfrac{1}{2}\,\phi(y=-d).$$

Laplace equation + bottom impermeability:

$$\phi(x,y) \; = \; \tfrac{1}{2}\,\Phi(x+\mathrm{i}(y+d)) \; + \; \tfrac{1}{2}\,\Phi(x-\mathrm{i}(y+d)).$$

Application

If $\Phi \propto \tanh(kx)$:

$$\begin{aligned}
\phi \; &= \; \tfrac{1}{2}\,\tanh(kx + \mathrm{i}k(y+d)) \; + \; \tfrac{1}{2}\,\tanh(kx - \mathrm{i}k(y+d)) \\
&= \frac{\sinh kx\, \cosh kx}{\sinh^2 kx \; + \; \cos^2 k(y+d)} \\
&= \frac{\tanh kx}{\tanh^2 kx \; + \; \mathrm{sech}^2\, kx\, \cos^2 k(y+d)}
\end{aligned}$$