Lagrangian Description of Steady Surface Waves

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Plan

- Statement of the problem
- Inadequacy of Stokes' approximation
- Eulerian & Lagrangian equations
- General steady flow
- Lagrangian Stokes-like expansion
- Examples

Eulerian vs Lagrangian

- Eulerian description:
  Observation at fixed positions x, y

- Lagrangian description:
  Follow particles from their initial positions x_0, y_0
  or related functions $\alpha = \alpha(x_0, y_0)$ $\beta = \beta(x_0, y_0)$

Eulerian linear irrotational waves

- Deep water case (incompressible fluid)

$$
\begin{aligned}
k\,\eta &= \varepsilon\,\cos k(x-ct) \\
(k/c)\,\phi &= \varepsilon\,\mathrm{e}^{ky}\sin k(x-ct) \\
c &= \sqrt{g/k}
\end{aligned}
$$

x, y : horizontal and upward-vertical coordinates;
t : time;
η : free surface;
ϕ : velocity potential;
g : acceleration due to gravity;
k : wavenumber;
c : phase velocity;
ε : steepness.

Higher-order approximations

- Second-order:

$$\begin{aligned} k\,\eta &= \varepsilon\ \cos\theta\ +\ \tfrac{1}{2}\varepsilon^2\ \cos 2\theta \\ \sqrt{k^3/g}\ \phi &= \varepsilon\,\mathrm{e}^{ky}\sin\theta \\ \theta &= k\left(x - t\sqrt{g/k}\right) \end{aligned}$$

Higher-order approximations

- Second-order:

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k\,\eta &= \varepsilon\ \cos\theta \ +\ \tfrac{1}{2}\varepsilon^2\ \cos 2\theta \\
\sqrt{k^3/g}\ \phi &= \varepsilon\,\mathrm{e}^{ky}\sin\theta \\
\theta &= k\left(x - t\sqrt{g/k}\right)
\end{aligned}
$$

- Third-order:

$$
\begin{aligned}
k\,\eta &= (\varepsilon - \tfrac{3}{8}\varepsilon^3)\cos\theta \ +\ \tfrac{1}{2}\varepsilon^2\cos 2\theta \ +\ \tfrac{3}{8}\varepsilon^3\cos 3\theta \\
&\quad + \tfrac{1}{2}\varepsilon^3\,\boldsymbol{t}\ \sin\theta \\
\sqrt{k^3/g}\ \phi &= (\varepsilon - \tfrac{1}{2}\varepsilon^3)\,\mathrm{e}^{ky}\sin\theta \ -\ \tfrac{1}{2}\varepsilon^3\,\boldsymbol{t}\,\mathrm{e}^{ky}\cos\theta
\end{aligned}
$$

Stokes' Eulerian expansion

- Power series expansion

$$\eta = \varepsilon\eta_1 + \varepsilon^2\eta_2 + \varepsilon^3\eta_3 + \cdots$$
$$\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \cdots$$
$$c = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \cdots$$

- Third-order approximation

$$k\,\eta = (\varepsilon - \tfrac{3}{8}\varepsilon^3)\cos\vartheta + \tfrac{1}{2}\varepsilon^2\cos 2\vartheta + \tfrac{3}{8}\varepsilon^3\cos 3\vartheta$$
$$(k/c)\,\phi = (\varepsilon - \tfrac{1}{2}\varepsilon^3)\,\mathrm{e}^{ky}\sin\vartheta$$
$$\vartheta = x - ct$$
$$c = \sqrt{g/k}\,\left(1 + \tfrac{1}{2}\varepsilon^2\right)$$

Stokes' alternative expansion

- 2D potential flows + incompressible fluids.
- Velocity potential ϕ and stream function ψ as independent variables:

$\Rightarrow$ Conformal mapping onto a strip

- Simpler algebra.
- Low rate of convergence.

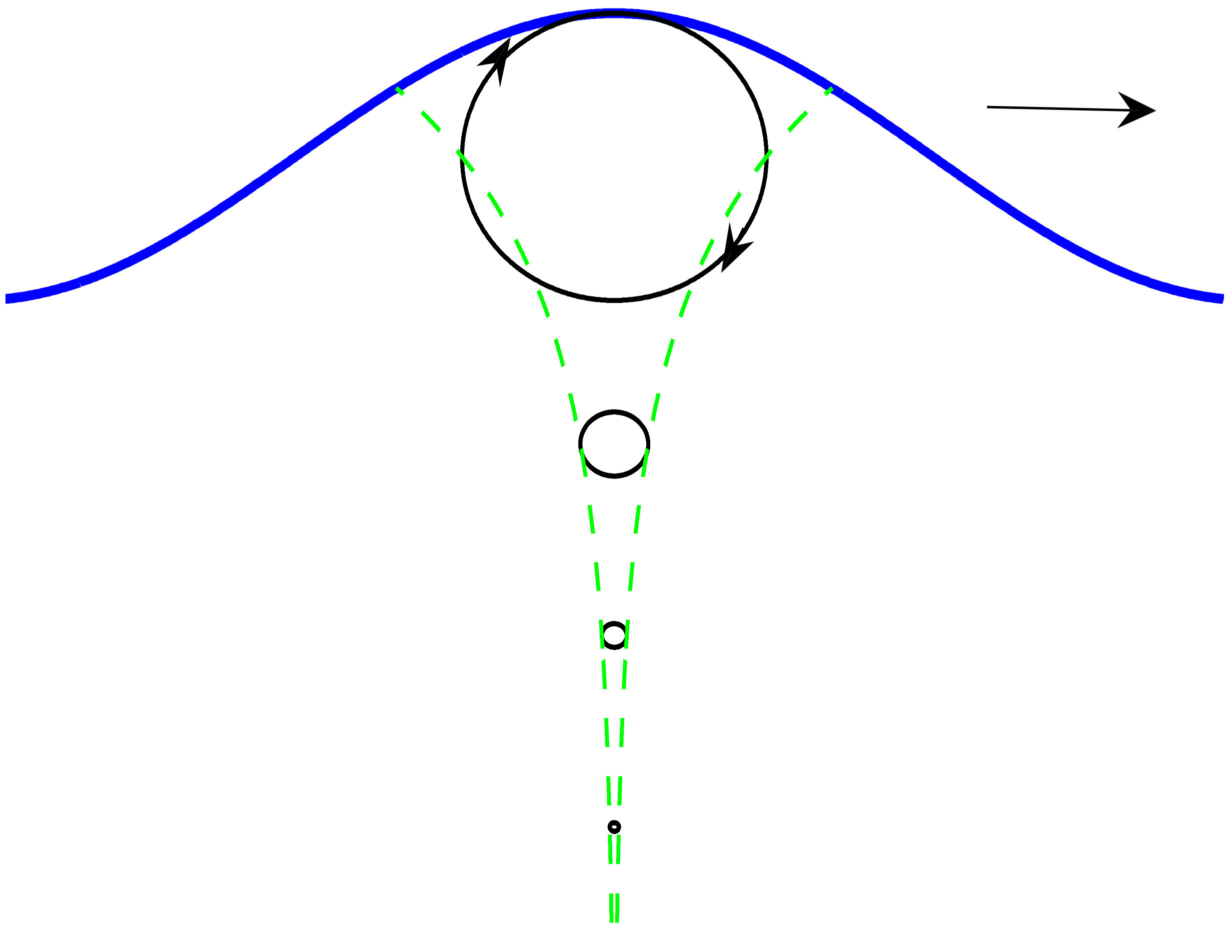
Lagrangian linear irrotational waves

- Deep water case

$$kx = k\alpha - \varepsilon\, e^{k\beta} \sin k(\alpha - ct)$$
$$ky = k\beta + \varepsilon\, e^{k\beta} \cos k(\alpha - ct)$$

x, y : particle's position
α, β : particle's labels (functions of x_0 and y_0)
t : time;
k : wavenumber;
c : phase velocity;
ε : steepness.

Trajectories of linear irrotational waves



Stokes' drift

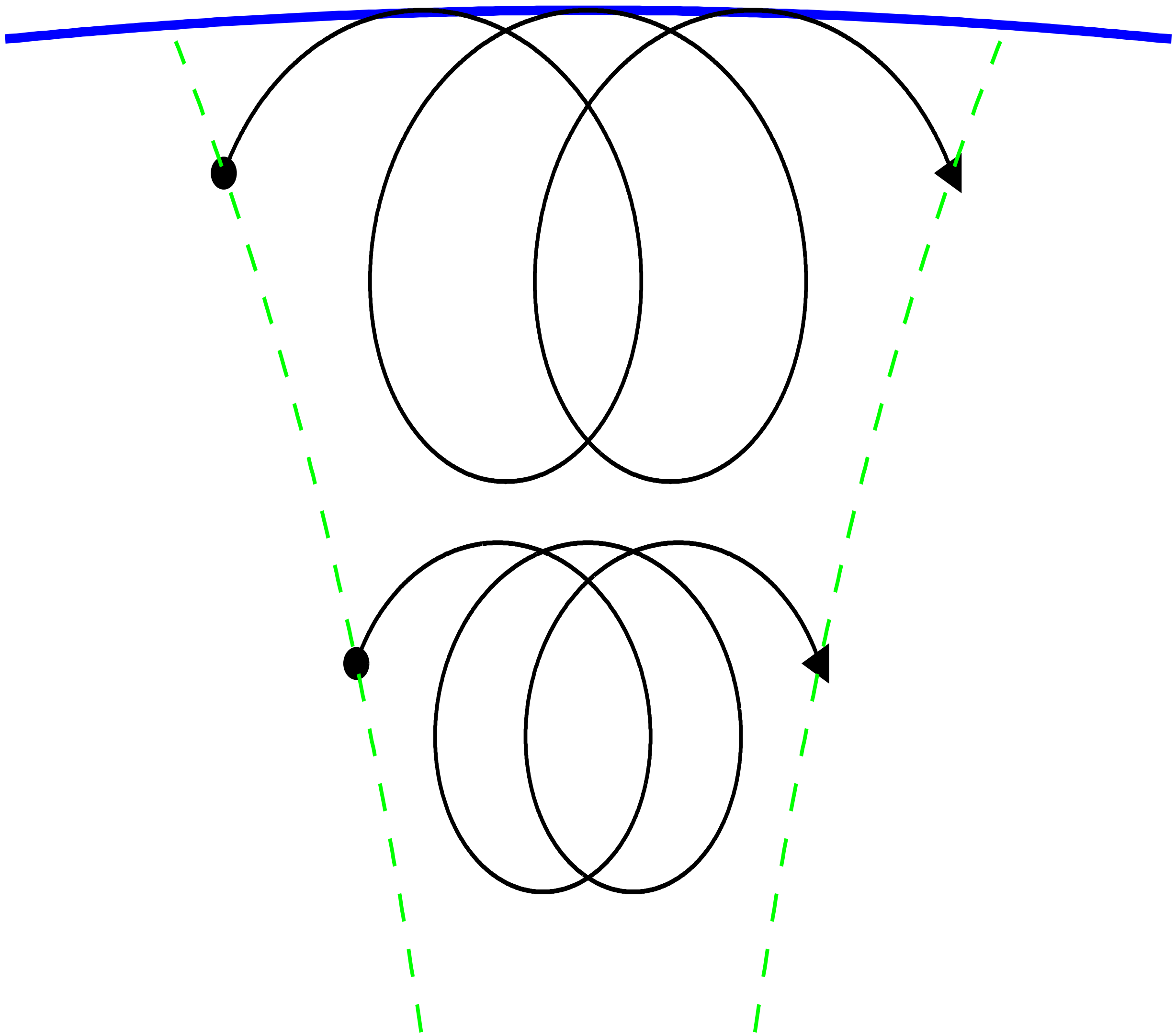
- Second-order Stokes' approximation for irrotational wave in deep water:

$$kx = k\alpha + \varepsilon^2 ct\,\mathrm{e}^{2k\beta} - \varepsilon\,\mathrm{e}^{k\beta}\sin k(\alpha - ct)$$
$$ky = k\beta + \tfrac{1}{2}\varepsilon^2 + \varepsilon\,\mathrm{e}^{k\beta}\cos k(\alpha - ct)$$

- Stokes' drift velocity

$$V = \varepsilon^2\,c\,\mathrm{e}^{2k\beta}$$

Trajectories with drift



Formal solutions

- All approximations are on the form:

$$x \;=\; \alpha \;+\; \gamma(\beta)\,t \;+\; X(\alpha - ct, \beta)$$

$$y \;=\; \beta \;+\; Y(\alpha - ct, \beta)$$

X, Y: Fourier polynomials.

Third-order Stokes' approximation

- Deep water :

$$
\begin{aligned}
kx &= k\alpha + \varepsilon^2 ct\,\mathrm{e}^{2k\beta} - \varepsilon\,\mathrm{e}^{k\beta}\sin k\theta \\
&\quad + \varepsilon^3\,\mathrm{e}^{3k\beta}\left[\left(\tfrac{3}{2}\mathrm{e}^{-2k\beta} - \tfrac{5}{2}\right)\sin k\theta + \boldsymbol{\theta}\cos k\theta\right]
\end{aligned}
$$

$$
\begin{aligned}
ky &= k\beta + \tfrac{1}{2}\varepsilon^2 + \varepsilon\,\mathrm{e}^{k\beta}\cos k(\alpha - ct) \\
&\quad - \varepsilon^3\,\mathrm{e}^{3k\beta}\left[\left(\tfrac{3}{2}\mathrm{e}^{2k\beta} - \tfrac{3}{2}\right)\cos k\theta - \boldsymbol{\theta}\sin k\theta\right]
\end{aligned}
$$

$$\theta = \alpha - ct$$

$$c = \sqrt{g/k}\,\left(1 + \tfrac{1}{2}\varepsilon^2\right)$$

Problems with the Stokes theory

- Secular terms (unbounded solutions).

Problems with the Stokes theory

- Secular terms (unbounded solutions).
- Impossibility to recast the solution as

$$x \;=\; \alpha \;+\; \gamma(\alpha,\beta)\,t \;+\; X(\alpha - ct, \beta)$$
$$y \;=\; \beta \;+\; Y(\alpha - ct, \beta)$$

Problems with the Stokes theory

- Secular terms (unbounded solutions).
- Impossibility to recast the solution as

$$x = \alpha + \gamma(\alpha, \beta)\, t + X(\alpha - ct, \beta)$$
$$y = \beta + Y(\alpha - ct, \beta)$$

- Question:
  What is the general Lagrangian mathematical expression of a steady wave?

Steady 2D Eulerian formulation

- Incompressibility

$$u \;=\; \frac{\mathrm{D}x}{\mathrm{D}t} \;=\; \frac{\partial\psi}{\partial y} \qquad\qquad v \;=\; \frac{\mathrm{D}y}{\mathrm{D}t} \;=\; -\frac{\partial\psi}{\partial x}$$

- Steadiness

$$\frac{\partial\psi}{\partial t} \;=\; 0$$

- Vorticity

$$\frac{\partial v}{\partial x} \;-\; \frac{\partial u}{\partial y} \;=\; \omega(\psi)$$

- Momentum

$$2P \;+\; 2\,g\,y \;+\; u^2 \;+\; v^2 \;=\; B(\psi)$$

Lagrangian formulation

- Particle position (x_0, y_0: initial position)

$$x \;=\; x(x_0, y_0, t) \qquad\qquad y \;=\; y(x_0, y_0, t)$$

- Incompressibility

$$\frac{\partial(x, y)}{\partial(x_0, y_0)} \;=\; 1$$

- Vorticity

$$\omega(\psi) \;=\; \frac{\partial(x_t, x)}{\partial(x_0, y_0)} \;+\; \frac{\partial(y_t, y)}{\partial(x_0, y_0)}$$

- Momentum

$$2P \;+\; 2\,g\,y \;+\; x_t^2 \;+\; y_t^2 \;=\; B(\psi)$$

Inconvenience of initial positions

- Domain (x_0, y_0) unknown
- Waves are not functions of $x_0 - ct$, e.g.

$$x = x(x_0 - ct, y_0)$$

at the surface $y_0 = \eta(x_0)$

$$x = x(x_0 - ct, \eta(x_0))$$

$\Rightarrow$ function of $x_0 - ct$ only if $\eta = 0$.

Parametric variables

- Mapping $(x_0, y_0) \mapsto (\alpha, \beta)$ onto a strip
  - $\beta = 0$: free surface
  - $\beta = -d$: bottom

$$\frac{\partial(x_0, y_0)}{\partial(\alpha, \beta)} \;=\; J(\alpha, \beta) \;>\; 0$$

- Incompressiblity + irrotationality

$$\frac{\partial(x, y)}{\partial(\alpha, \beta)} \;=\; J \qquad\qquad \frac{\partial(x_t, x)}{\partial(\alpha, \beta)} + \frac{\partial(y_t, y)}{\partial(\alpha, \beta)} \;=\; J\,\omega$$

Example: Gerstner's wave

- Gerstner's exact rotational solution

$$
\begin{aligned}
x &= \alpha - ct \; - \; a\,\mathrm{e}^{k\beta}\sin k(\alpha - ct) \\
y &= \beta \; + \; \tfrac{1}{2}ka^2 \; + \; a\,\mathrm{e}^{k\beta}\cos k(\alpha - ct) \\
J &= 1 \; - \; (ka)^2\,\mathrm{e}^{2k\beta} \\
\omega &= \frac{2k^3c\,a^2\,\mathrm{e}^{2k\beta}}{1 - (ka)^2\,\mathrm{e}^{2k\beta}} \\
c &= \sqrt{g\,/\,k}
\end{aligned}
$$

Lagrangian stream function

By definition $\psi = \psi(x, y)$ hence

$$\begin{aligned} \mathrm{d}\psi &= u\,\mathrm{d}y \;-\; v\,\mathrm{d}x \\ &= (u\,y_\alpha - v\,x_\alpha)\,\mathrm{d}\alpha \;+\; (u\,y_\beta - v\,x_\beta)\,\mathrm{d}\beta \\ &\quad + (u\,y_t - v\,x_t)\,\mathrm{d}t \end{aligned}$$

thus $\psi = \psi(\alpha, \beta)$ and

$$\frac{\partial \psi}{\partial \alpha} = \frac{\partial(x, y)}{\partial(t, \alpha)} \qquad \frac{\partial \psi}{\partial \beta} = \frac{\partial(x, y)}{\partial(t, \beta)}$$

Lagrangian steady flow

From ψ's definition

$$u = \frac{\partial\,\psi}{\partial y} = \frac{\partial(x,\psi)}{\partial(x,y)} = \frac{\partial(x,\psi)}{\partial(\alpha,\beta)}\,\frac{\partial(\alpha,\beta)}{\partial(x,y)}$$

$$\Longrightarrow \qquad \frac{\partial\,x}{\partial t} = \frac{1}{J}\,\frac{\partial(x,\psi)}{\partial(\alpha,\beta)}$$

hence

$$x = x(\tau,\psi) \qquad \tau \equiv t + T(\alpha,\beta) \qquad \frac{\partial(T,\psi)}{\partial(\alpha,\beta)} = J$$

Special coordinates

- $\psi = \psi(\beta)$

$$x \;=\; x\,(\tau, \beta) \qquad\qquad \tau \;=\; t \;+\; \left(\frac{\mathrm{d}\,\psi}{\mathrm{d}\beta}\right)^{-1} \int J \,\mathrm{d}\alpha$$

- $\psi = \psi(\beta)$ and $J = J(\beta)$ (Abrashkin *et al.* 1990; Chang *et al.* 2007)

$$x \;=\; x\,(\alpha - C(\beta)t, \beta)$$

- $\psi = -c\beta$ and $J = 1$

$$x \;=\; x\,(\alpha - ct, \beta)$$

Stokes-like expansion I

- Fourier series simplified coordinates

$$
\begin{aligned}
kx &= K\xi - \sum_{n=1}^{\infty} kX_n \sin nK\xi \\
ky &= \sum_{n=0}^{\infty} kY_n \cos nK\xi \\
\xi &= \alpha - C(\beta)\, t
\end{aligned}
$$

$K(\beta)$: apparent wavenumber
$C(\beta)$: apparent phase velocity

- Drift velocity

$$V = c - C\, K \,/\, k$$

Stokes-like expansion II

- Small parameter expansion

$$\{X_n\,;\,Y_n\} \;=\; \sum_{j=0}^{\infty} \varepsilon^{n+j}\,\{X_{n,j}\,;\,Y_{n,j}\}$$

$$\{K\,;\,C\} \;=\; \sum_{j=0}^{\infty} \varepsilon^{j}\,\{K_j\,;\,C_j\}$$

- Steepness definition

$$\varepsilon \;=\; \sum_{n=1}^{\infty}\sum_{j=0}^{\infty} \varepsilon^{2n+j-1}\,k\,Y_{2n-1,j}(\beta=0)$$

Deep water irrotational wave

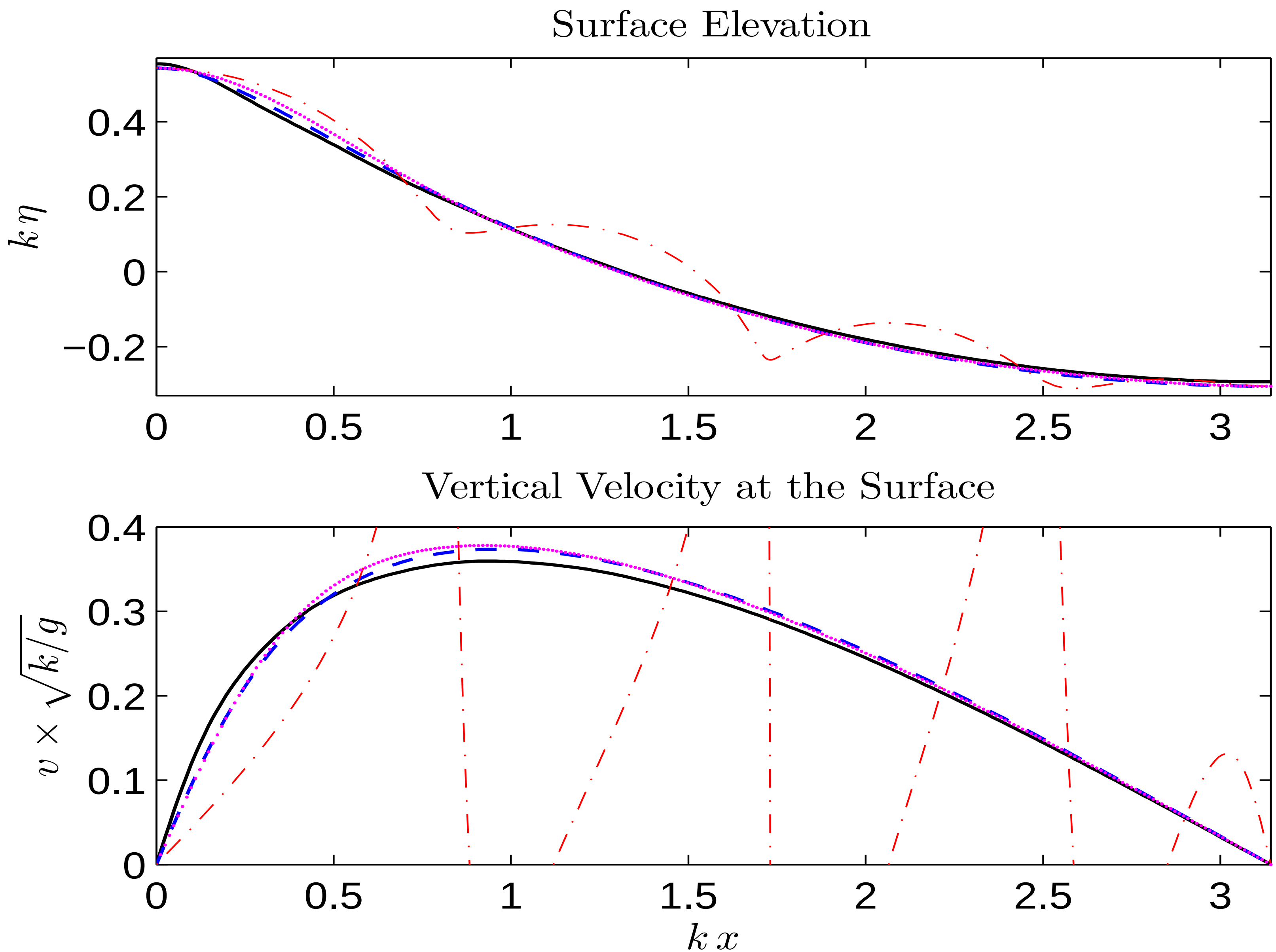
- Third-order approximation in normalized coordinates

$$
\begin{aligned}
kx &= K\xi \;-\; \varepsilon\,\mathrm{e}^{k\beta}\sin K\xi \;+\; \varepsilon^3\left(\tfrac{3}{2}\mathrm{e}^{k\beta} - \tfrac{5}{2}\mathrm{e}^{3k\beta}\right)\sin K\xi \\
ky &= k\beta \;+\; \varepsilon\,\mathrm{e}^{k\beta}\cos K\xi \;+\; \varepsilon^2\left(\mathrm{e}^{2k\beta} - \tfrac{1}{2}\right) \\
&\quad -\; \varepsilon^3\left(\tfrac{3}{2}\mathrm{e}^{k\beta} - \tfrac{3}{2}\mathrm{e}^{3k\beta}\right)\cos K\xi \\
K &= k\left(1 - \varepsilon^2\mathrm{e}^{2k\beta}\right) \\
c &= \sqrt{g/k}\,\left(1 + \tfrac{1}{2}\varepsilon^2\right)
\end{aligned}
$$

- Drift velocity

$$
\frac{V}{c} = \left(\varepsilon^2 - 3\varepsilon^4 + \tfrac{13}{6}\varepsilon^6\right)\mathrm{e}^{2k\beta} + \left(3\varepsilon^4 - 14\varepsilon^6\right)\mathrm{e}^{4k\beta} + \tfrac{53}{4}\varepsilon^6\,\mathrm{e}^{6k\beta}
$$

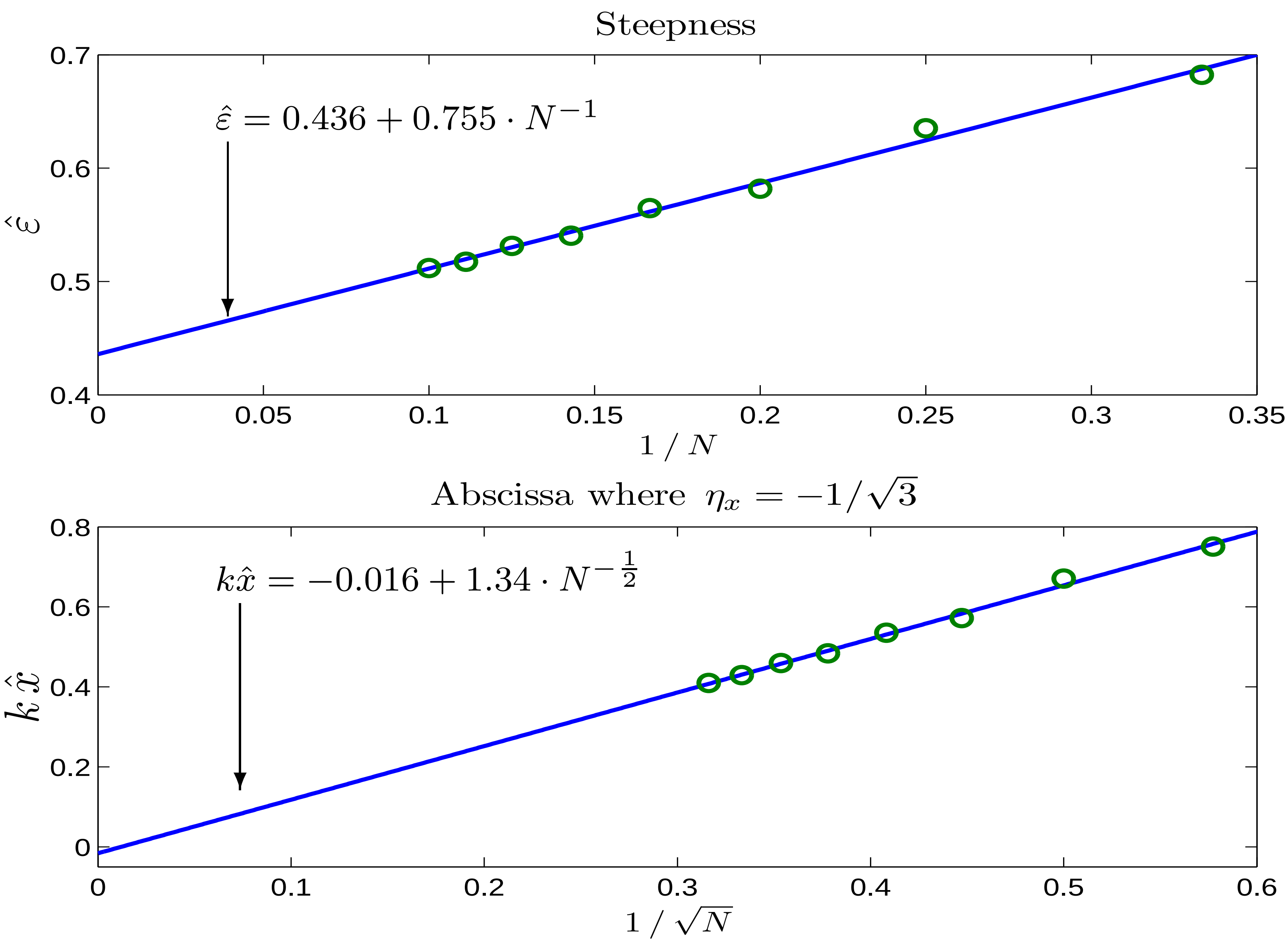
Comparison with other approximations



Deep water seventh-order approximations and exact solution ($\omega = 0$, $\varepsilon = 0.424$):

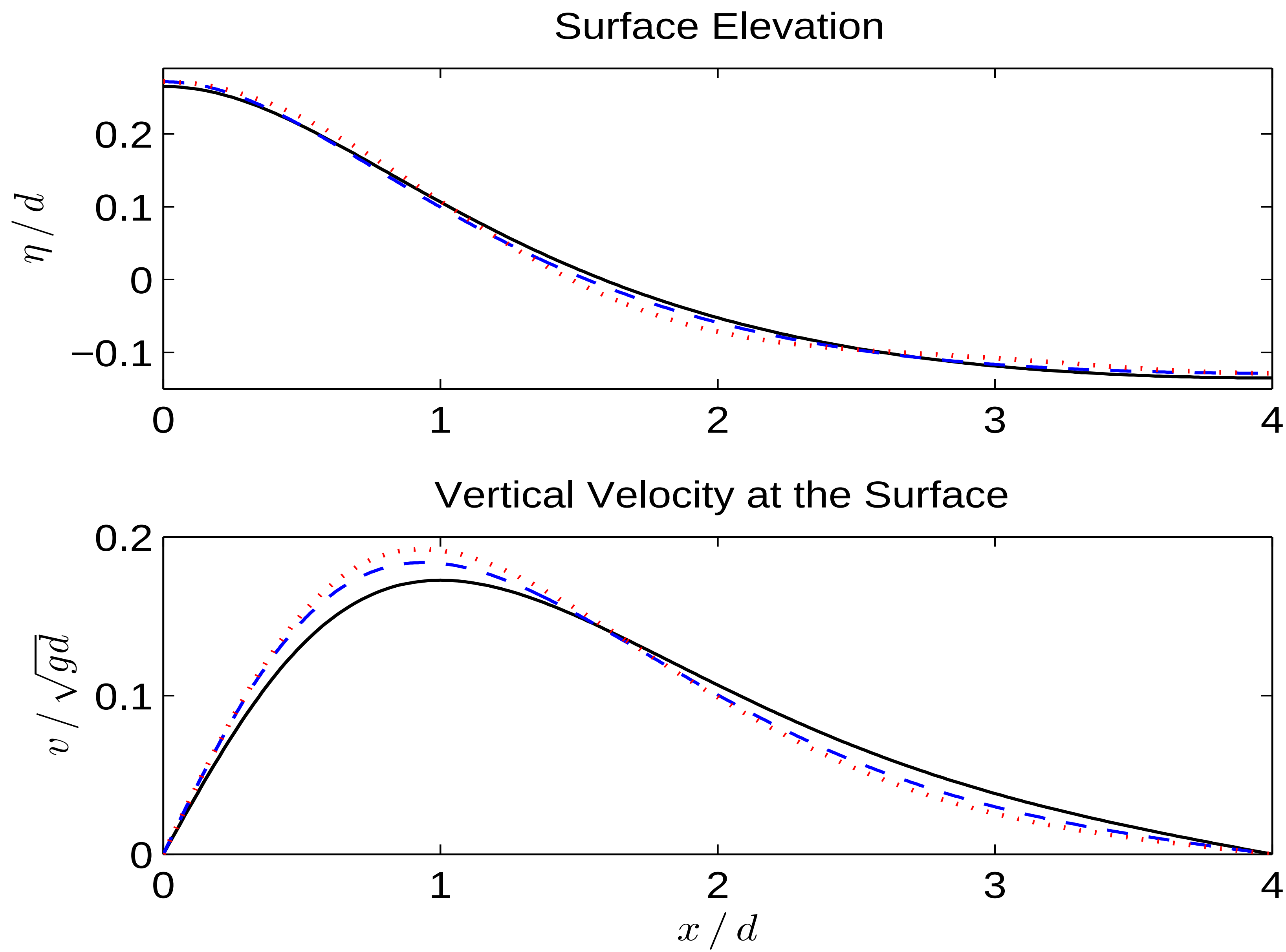
—, exact (Fenton 1988); – –, Lagrange; $\cdots$, Euler; – ·, Stokes.

Highest wave



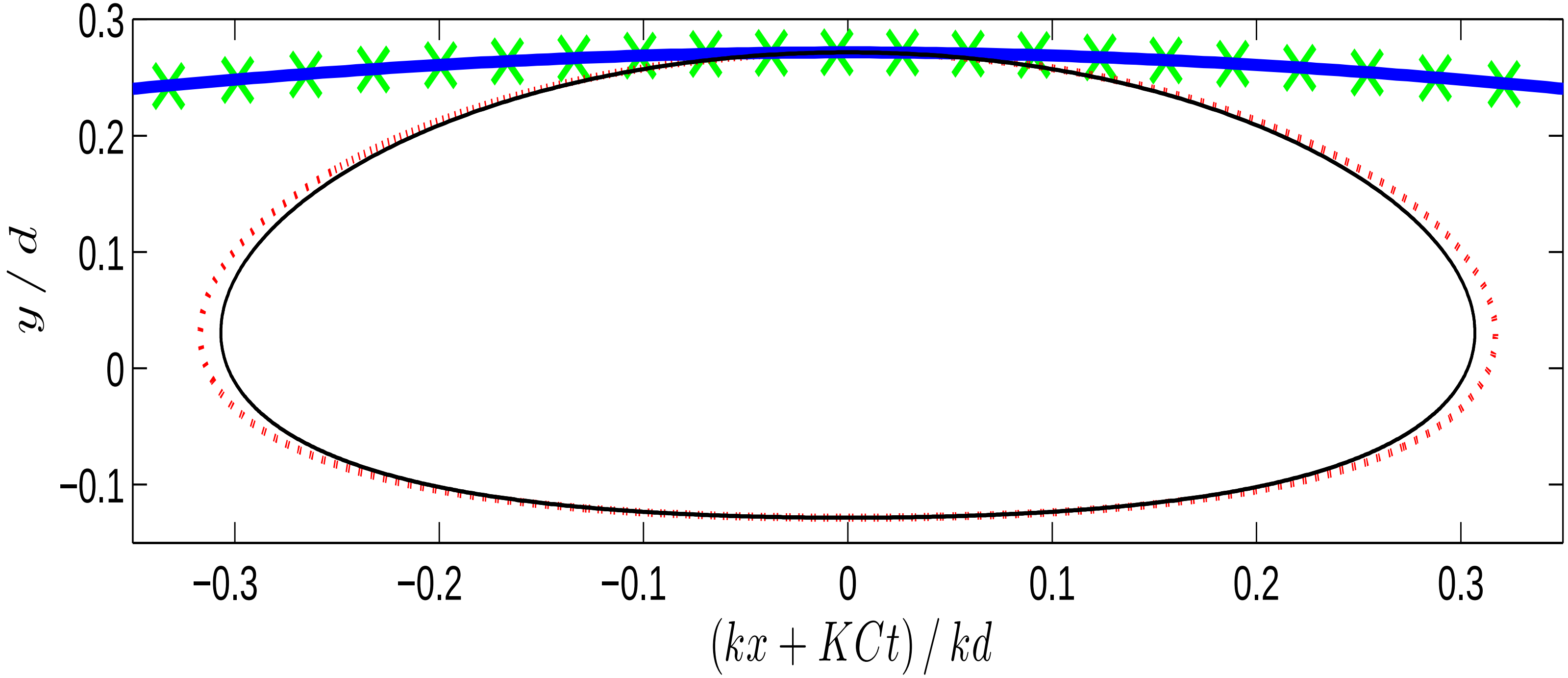
○, third- to tenth-order approximations; —, linear regressions.

Finite depth irrotational wave



—, exact (Fenton 1988); – –, 3rd-order Lagrange; · · · , 3rd-order Euler.

Finite depth Gerstner wave



▬, Gerstner surface; $\times$, irrotational surface; —, Gerstner trajectory; $\cdots$ irrotational trajectories after substraction of the Stokes drift.

Conclusion

- Definition of Lagrangian steady flows
- Stokes-like theory without secular terms
- Improved approximations
- Possible generalization for 3D compressible fluids

Futur studies

- Convergence for steep waves
- Standing waves
- Other waves