Water waves determination from pressure measurements

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Motivation

Goal:

- Measurement of the surface water waves

Constraints:

- Non-intrusive probes
- Easy maintenance

Solution:

- Pressure gauges at the bottom
- Surface recovery from pressure data

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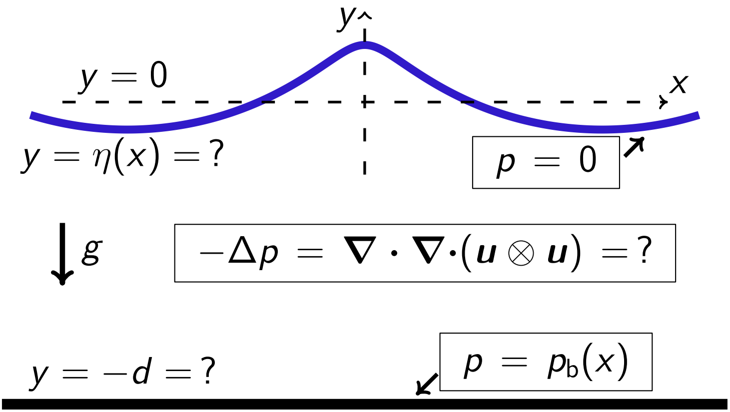
Problem:

- How?

Typical Problem



Model Problem

Hypothesis:

- Two-dimensional problem
- Permanent flow
- Perfect fluid with constant density
- Irrotational motion
- Flat horizontal bottom

Notations:

- d: mean depth
- g: acceleration due to gravity
- p: pressure divided by density
- x, y: horizontal and upward vertical coordinates
- u, v: horizontal and vertical velocities
- ϕ, ψ: velocity potential and stream function

Fundamental Equations

Incompressibility and Irrotationality for $x \in \mathbb{R}$, $y \in [-d, \eta]$:

$$u \;=\; \partial_x \phi \;=\; \partial_y \psi, \qquad v \;=\; \partial_y \phi \;=\; -\partial_x \psi.$$

Bottom's impermeability at $y = -d$:

$$v_{\mathrm{b}} \;=\; 0.$$

Free surface's impermeability at $y = \eta(x)$:

$$v_{\mathrm{s}} \;=\; \eta_x \, u_{\mathrm{s}}$$

Free surface isobarity at $y = \eta(x)$:

$$2\, g\, \eta \;+\; u_{\mathrm{s}}^{2} \;+\; v_{\mathrm{s}}^{2} \;=\; B$$

Subscripts 'b' and 's' denote quantities at the bottom and at the surface, and $\eta_x \,=\, \mathrm{d}\eta \,/\, \mathrm{d}x$

Definition of Physical Quantities

- **$(2\pi/k)$-periodic waves ($k \to 0$ for solitary waves).**
- **Definition of the mean water level:**

$$\langle\, \eta \,\rangle \;\equiv\; \frac{k}{2\pi} \int_{-\pi/k}^{\pi/k} \eta(x)\, \mathrm{d}x \;=\; 0.$$

- **Bernoulli's constant:**

$$B \;=\; \left\langle\, {u_\mathsf{s}}^2 \,+\, {v_\mathsf{s}}^2 \,\right\rangle \;=\; \left\langle\, {u_\mathsf{b}}^2 \,\right\rangle \;>\; 0.$$

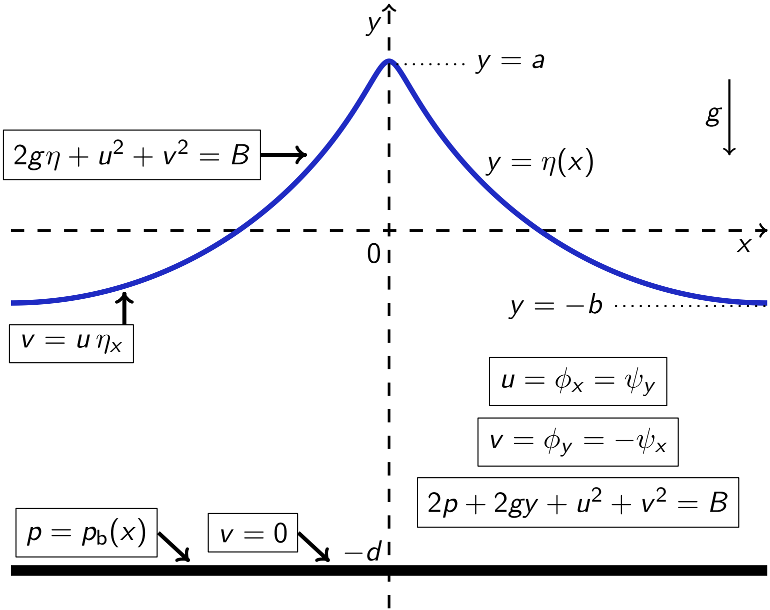
- **Mean pressure at the bed:**

$$\langle\, p_\mathsf{b} \,\rangle \;=\; g\, d.$$

- **Bernoulli's equation:**

$$2\, p \;+\; 2\, g\, y \;+\; u^2 \;+\; v^2 \;=\; B.$$

Definition Sketch



Linear Wave

Approximation for infinitesimal waves in finite depth:

$$\begin{aligned}
\eta(x) &\approx a\cos(kx)\\
\phi(x,y) &\approx -c\,x + \frac{c\,a}{\sinh(kd)}\cosh(k(y+d))\sin(kx)\\
p(x,y) &\approx -g\,y + \frac{c^2\,k\,a}{\sinh(kd)}\cosh(k(y+d))\cos(kx)\\
B &\approx c^2\\
c^2 &\approx g\,k\,\tanh(kd)
\end{aligned}$$

Hydrostatic Recovery

Hydrostatic approximation:

$$p(x, y) \approx g\,(\,\eta(x) - y\,).$$

Approximate surface:

$$\eta(x) \approx p_{\mathrm{b}}(x)\,/\,g - d.$$

Error exceeding 15% for moderate waves.
(Bishop & Donelan 1987, *Coastal Engineering* **11**)

Linear Wave Recovery

Considering the linear wave approximation:

$$\eta(x) \approx \cosh(kd)\,[\,p_{\mathrm{b}}(x)\,/\,g - d\,]\,.$$

(Escher & Schlurmann 2008, *J. Nonlinear Math. Phys.* **15**)

Overestimate large waves height by more than 10%.
(Tsai *et al.* 2005, *Ocean Engineering* **32**)

Fully Nonlinear Recovery

Without approximations:

$$\sqrt{\frac{B\,-\,2\,g\,\eta(x)}{1\,+\,[\,\eta_x(x)\,]^2}}\;=\;\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}nkx}\,\hat{\theta}_n\,\cosh(nk[\eta(x)+d])\,,$$

where

$$\hat{\theta}_n\;=\;\frac{k}{2\pi}\int_0^{2\pi/k}\mathrm{e}^{-\mathrm{i}nkx}\sqrt{B\,-\,2\,p_{\mathrm{b}}(x)\,+\,2\,g\,d}\;\mathrm{d}x.$$

(Oliveras *et al.* 2012, *SIAM J. Appl. Math.* **72**, **3**)

Complex Variables

Complex abscissa:

$$z \;=\; x \;+\; \mathrm{i}\, y$$

Complex potential:

$$f(z) \;=\; \phi(x,y) \;+\; \mathrm{i}\,\psi(x,y)$$

Complex velocity:

$$w(z) \;=\; u(x,y) \;-\; \mathrm{i}\, v(x,y) \;=\; \frac{\mathrm{d}\, f}{\mathrm{d}z}$$

Bernoulli equation:

$$p(z,\bar{z}) \;=\; \frac{B}{2} \;-\; \frac{g\,(z-\bar{z})}{2\,\mathrm{i}} \;-\; \frac{1}{2}\left|\frac{\mathrm{d}\, f}{\mathrm{d}z}\right|^2$$

Complex Pressure

Let be the holomorphic function:

$$\mathfrak{P}(z) \;\equiv \tfrac{1}{2}\,B \;+\; g\,d \;-\; \tfrac{1}{2}\,w^2(z)$$

Complex Pressure

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$$\begin{aligned}\mathfrak{P}(z) &\equiv \tfrac{1}{2}\,B \;+\; g\,d \;-\; \tfrac{1}{2}\,w^2(z) \\ &= \tfrac{1}{2}\,B \;+\; g\,d \;-\; \tfrac{1}{2}\,(u^2 - v^2) \;+\; \mathrm{i}\,u\,v\end{aligned}$$

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At the bottom $z = x - \mathrm{i}d$ **where** $v = 0$**:**

$$\mathfrak{P}(x - \mathrm{i}d) \ = \ \tfrac{1}{2}\, B \ + \ g\, d \ - \ \tfrac{1}{2}\, u_{\mathrm{b}}^2 \ = \ p_{\mathrm{b}}(x)$$

In the fluid $x \in \mathbb{R}\,,\ y \in [-d, \eta(x)]$**:**

$$\mathfrak{P}(z) \ = \ p_{\mathrm{b}}(z + \mathrm{i}d)$$

Note that $p \neq \mathrm{Re}\{\mathfrak{P}\}$ **and** $p \neq \mathrm{Im}\{\mathfrak{P}\}$ **if** $y \neq -d$**.**
In particular, $\mathfrak{P} \neq 0$ **at** $y = \eta(x)$**.**

Complex Surface Boundary Condition

At the free surface $y = \eta(x)$ **where** $v_s = \eta_x u_s$**:**

$$w_s^{\,2} \;=\; (\,u_s \;-\; \mathrm{i}\,v_s\,)^2 \;=\; (\,1 \;-\; \mathrm{i}\,\eta_x\,)^2 \; u_s^2$$

Complex Surface Boundary Condition

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$$
\begin{aligned}
{w_s}^2 &= (\, u_s - \mathrm{i}\, v_s \,)^2 = (\, 1 - \mathrm{i}\, \eta_x \,)^2 \; {u_s}^2 \\
&= (\, 1 - \mathrm{i}\, \eta_x \,)^2 \; u_s^2 \,(\, 1 + \mathrm{i}\, \eta_x \,) \,/\, (\, 1 + \mathrm{i}\, \eta_x \,)
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&= \left(1 + \eta_x^2\right) u_s^2\,(\,1 - \mathrm{i}\,\eta_x\,)\,/\,(\,1 + \mathrm{i}\,\eta_x\,) \\
&= \left(\,{u_s}^2 + {v_s}^2\,\right)(\,1 - \mathrm{i}\,\eta_x\,)\,/\,(\,1 + \mathrm{i}\,\eta_x\,)
\end{aligned}
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Complex Surface Boundary Condition

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\begin{aligned}
{w_s}^2 &= (\, u_s - \mathrm{i}\, v_s \,)^2 = (\, 1 - \mathrm{i}\, \eta_x \,)^2 \, u_s^2 \\
&= (\, 1 - \mathrm{i}\, \eta_x \,)^2 \, u_s^2 \, (\, 1 + \mathrm{i}\, \eta_x \,) \, / \, (\, 1 + \mathrm{i}\, \eta_x \,) \\
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&= \left(u_s^2 + {v_s}^2 \right) (\, 1 - \mathrm{i}\, \eta_x \,) \, / \, (\, 1 + \mathrm{i}\, \eta_x \,) \\
&= (\, B - 2\, g\, \eta \,) (\, 1 - \mathrm{i}\, \eta_x \,) \, / \, (\, 1 + \mathrm{i}\, \eta_x \,)
\end{aligned}
$$

Complex Surface Boundary Condition

At the free surface $y = \eta(x)$ **where** $v_s = \eta_x u_s$**:**

$$\begin{aligned} {w_s}^2 &= (\, u_s - \mathrm{i}\, v_s \,)^2 = (\, 1 - \mathrm{i}\, \eta_x \,)^2 \, u_s^2 \\ &= (\, 1 - \mathrm{i}\, \eta_x \,)^2 \, u_s^2 \, (\, 1 + \mathrm{i}\, \eta_x \,) \,/\, (\, 1 + \mathrm{i}\, \eta_x \,) \\ &= \left(1 + \eta_x^2\right) u_s^2 \, (\, 1 - \mathrm{i}\, \eta_x \,) \,/\, (\, 1 + \mathrm{i}\, \eta_x \,) \\ &= \left(u_s^2 + {v_s}^2\right) (\, 1 - \mathrm{i}\, \eta_x \,) \,/\, (\, 1 + \mathrm{i}\, \eta_x \,) \\ &= (\, B - 2\, g\, \eta \,) (\, 1 - \mathrm{i}\, \eta_x \,) \,/\, (\, 1 + \mathrm{i}\, \eta_x \,) \end{aligned}$$

Thence, after multiplication by $1 + \mathrm{i}\eta_x$ **and exploiting** $(B - {w_s}^2) = 2(\mathfrak{P}_s - gd)$ **with** $\mathfrak{P}_s(x) = \mathfrak{P}(x + \mathrm{i}\eta(x))$:

$$g\, \eta \, (\, 1 - \mathrm{i}\, \eta_x \,) + \mathrm{i}\, B\, \eta_x = (\, \mathfrak{P}_s - g\, d \,) (\, 1 + \mathrm{i}\, \eta_x \,)$$

Real Surface Boundary Conditions

Splitting real and imaginary parts:

$$g\,\eta \;=\; \tfrac{1}{2}\left[\,B \;-\; u_{\mathrm{s}}^{\;2} \;-\; v_{\mathrm{s}}^{\;2}\,\right]$$
$$(B - g\,\eta)\,\eta_x \;=\; \tfrac{1}{2}\left[\,B \;+\; u_{\mathrm{s}}^{\;2} \;+\; v_{\mathrm{s}}^{\;2}\,\right]\eta_x$$

Both equations are ensured by the validity of the Bernoulli equation at the free surface.

Example 1: Fourier Series

Fitting pressure data with Fourier polynomials:

$$p_{\mathrm{b}}(x) \;\approx\; g\,d \;+\; \sum_{|n|>0}^{N} \mathfrak{p}_n \exp(\mathrm{i}nkx)$$

with $\mathfrak{p}_{-n} = \bar{\mathfrak{p}}_n$ **since** $p_{\mathbf{b}}$ **is real.**

Approximated complex pressure $(x \mapsto z + \mathrm{i}d)$**:**

$$\mathfrak{P}(z) \;=\; g\,d \;+\; \sum_{|n|>0}^{N} \mathfrak{p}_n \,\frac{\exp(\mathrm{i}nkz)}{\exp(nkd)},$$

Easy calculation!

Example 2: Cnoidal Wave

Fitting pressure data with Jacobian elliptic functions:

$$p_{\mathsf{b}}(x) \;\approx\; g\,d \;+\; A\left[\,\mathrm{dn}^2(\kappa x|m) \;-\; E(m)\,/\,K(m)\,\right]$$

Approximated complex pressure ($x \mapsto z + \mathrm{i}d$):

$$\mathfrak{P}(z) \;=\; g\,d \;+\; A\left[\,\mathrm{dn}^2(\kappa(z+\mathrm{i}d)|m) \;-\; E(m)\,/\,K(m)\,\right]$$

Solitary waves ($m \to 1$):

$$\mathfrak{P}(z) \;=\; g\,d \;+\; A\,\mathrm{sech}^2(\kappa(z+\mathrm{i}d))$$

Still very easy!

Equations for the Surface

Surface – Pressure equation:

$$g\,\eta\,(\,1\,-\,\mathrm{i}\,\eta_x\,)\,+\,\mathrm{i}\,B\,\eta_x\;=\;(\,\mathfrak{P}_\mathsf{s}\,-\,g\,d\,)\,(\,1\,+\,\mathrm{i}\,\eta_x\,) \qquad (1)$$

At the wave crest where $\eta_x = 0$, $x = 0$ and $\eta = a$:

$$a\;=\;g^{-1}\,\mathfrak{P}(\mathrm{i}a)\;-\;d$$

The imaginary part of (1) yields the ODE:

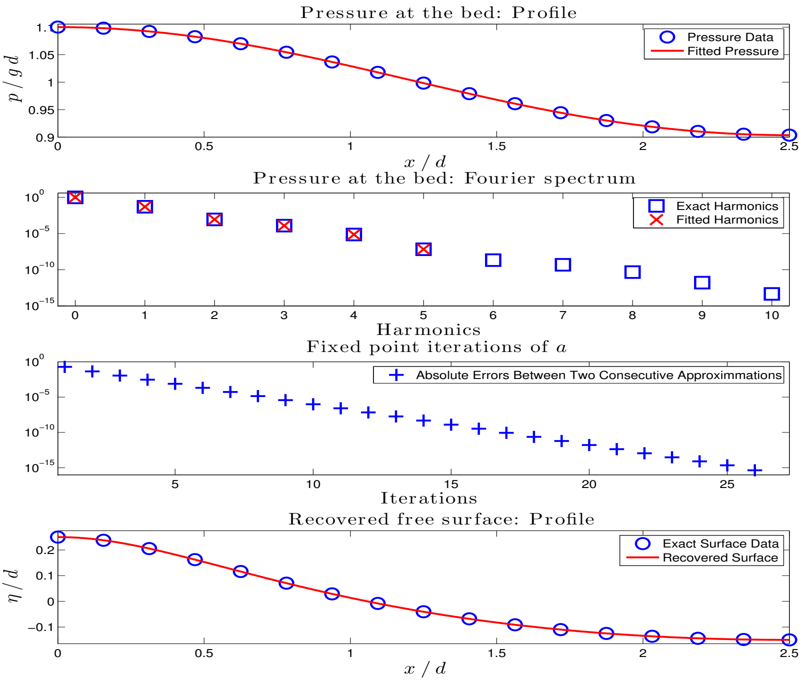
$$\eta_x\;=\;\mathrm{Im}\{\mathfrak{P}_\mathsf{s}\}\;/\;(\,B\;-\;g\,\eta\;-\;\mathrm{Re}\{\mathfrak{P}_\mathsf{s}\}\;+\;g\,d\,)$$

together with $\eta(0) = a$.

Bernoulli's constant B determined from $\langle\,\eta\,\rangle = 0$.

Application

- **Large wave in rather deep water:**
  - Wave Length / Depth: $2\pi/kd = 5$;
  - Wave Height / Depth: $(a+b)/d = 0.4$;
  - Amplitude / Depth: $a/d = 0.25$.
- **Exact solution computed numerically (Fenton 1988).**
- **Bottom pressure discretized at 32 equally spaced nodes.**
- **Fit of a fifth-order Fourier polynomial.**


Pressure at the bed: Profile
Pressure Data
Fitted Pressure
p / g d
x / d
Pressure at the bed: Fourier spectrum
Exact Harmonics
Fitted Harmonics
Harmonics
Fixed point iterations of a
Absolute Errors Between Two Consecutive Approximmations
Iterations
Recovered free surface: Profile
Exact Surface Data
Recovered Surface
η / d
x / d

Reconstruction Equations

At the wave crest where $\eta_x = 0$**,** $x = 0$ **and** $\eta = a$**:**

$$a \;=\; g^{-1}\,\mathfrak{P}(\mathrm{i}a) \;-\; d$$

ODE for the free surface:

$$\eta_x \;=\; \mathrm{Im}\{\mathfrak{P}_\mathrm{s}\} \;/\; (\,B \;-\; g\,\eta \;-\; \mathrm{Re}\{\mathfrak{P}_\mathrm{s}\} \;+\; g\,d\,)$$

together with $\eta(0) = a$**.**

Bernoulli's constant B **determined from** $\langle\,\eta\,\rangle = 0$**.**

(Clamond & Constantin 2013, *J. Fluid Mech.* **714**)

Analytic Resolution

Let be the new holomorphic function:

$$\mathfrak{Q}(z) \;\equiv\; \int_{z_0}^{z} \left[\, \mathfrak{P}(z') \,-\, g\, d \,\right] \mathrm{d}z' \;=\; \int_{z_0}^{z} \tfrac{1}{2} \left[\, B \,-\, w(z')^2 \,\right] \mathrm{d}z'$$

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At the free surface (with $z_0 = \mathrm{i}a$):

$$\begin{aligned} \mathfrak{Q}_{\mathsf{s}}(x) \;&=\; \int_0^{x} \left[\, \mathfrak{P}(x' + \mathrm{i}\eta(x')) \,-\, g\, d \,\right] \left[\, 1 \,+\, \mathrm{i}\, \eta_x(x') \,\right] \mathrm{d}x' \\ &=\; \int_0^{x} g\, \eta(x')\, \mathrm{d}x' \;+\; \mathrm{i} \left[\, \eta(x) \,-\, a \,\right] \left[\, B \,-\, \tfrac{1}{2}\, g\, a \,-\, \tfrac{1}{2}\, g\, \eta(x) \,\right] \end{aligned}$$

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Imaginary part:

$$\eta \;=\; g^{-1} \left[\, B \;-\; \sqrt{(\, B \,-\, g\, a\,)^2 \;-\; 2\, g\, \mathrm{Im}\{\mathfrak{Q}_{\mathsf{s}}\}} \,\right]$$

Example 1 Revisited

Fitting pressure data with Fourier polynomials:

$$p_{\mathrm{b}}(x) \;\approx\; g\,d \;+\; \sum_{|n|>0}^{N} \mathfrak{p}_n \exp(\mathrm{i}nkx)$$

Approximated complex pressure ($x \mapsto z + \mathrm{i}d$):

$$\mathfrak{P}(z) \;=\; g\,d \;+\; \sum_{|n|>0}^{N} \mathfrak{p}_n \frac{\exp(\mathrm{i}nkz)}{\exp(nkd)}$$

Antiderivative of $\mathfrak{P} - gd$:

$$\mathfrak{Q}(z) \;=\; \sum_{|n|>0}^{N} \frac{\mathfrak{p}_n}{\mathrm{i}\,n\,k} \frac{\exp(\mathrm{i}nkz) \;-\; \exp(-nka)}{\exp(nkd)}$$

Easy calculation!

Example 2 Revisited

Fitting pressure data with Jacobian elliptic functions:

$$p_{\mathrm{b}}(x) \;\approx\; g\,d \;+\; A\left[\,\mathrm{dn}^2(\kappa x|m) \;-\; E(m)\,/\,K(m)\,\right]$$

Approximated complex pressure ($x \mapsto z + \mathrm{i}d$):

$$\mathfrak{P}(z) \;=\; g\,d \;+\; A\left[\,\mathrm{dn}^2(\kappa(z+\mathrm{i}d)|m) \;-\; E(m)\,/\,K(m)\,\right]$$

Antiderivative of $\mathfrak{P} - gd$:

$$\mathfrak{Q}(z) \;=\; (A/\kappa)\left[\,\mathrm{Z}(\kappa(z+\mathrm{i}d)|m) \;-\; \mathrm{Z}(\mathrm{i}\kappa(a+d)|m)\,\right]$$

Solitary waves ($m \to 1$):

$$\mathfrak{Q}(z) \;=\; (A/\kappa)\left[\,\tanh(\kappa(z+\mathrm{i}d)) \;-\; \mathrm{i}\tan(\kappa(a+d))\,\right]$$

Still very easy!

Crest, Trough and Bernoulli's Constant

Crest amplitude given implicitly ($\eta_x = 0$, $x = 0$, $\eta = a$):

$$a \;=\; g^{-1}\,\mathfrak{P}(\mathrm{i}a) \;-\; d$$

Trough amplitude given implicitly ($\eta_x = 0$, $x = \pi/k$, $\eta = -b$):

$$b \;=\; d \;-\; g^{-1}\,\mathfrak{P}(-\mathrm{i}b)$$

Bernoulli constant given explicitly:

$$B \;=\; \tfrac{1}{2}\,g\,(a-b) \;-\; (a+b)^{-1}\;\mathrm{Im}\{\mathfrak{Q}(\pi/k-\mathrm{i}b)\}$$

Convergence at the Crest

Let be the iterations for $x = 0$ **and** $y \in [-d, a]$**:**

$$y \; = \; F(y) \; \equiv \; g^{-1} \, \mathrm{Re}\{\mathfrak{P}(\mathrm{i}y)\} \; - \; d$$

Convergence occurs if $-1 < F_y < 1$ **with:**

$$F_y \; = \; 1 \; + \; p_y(0, y) \, / \, g \; = \; - \, u(0, y) \, u_y(0, y) \, / \, g$$

Since $-g \leqslant p_y < 0$ **under the crest, the inequality is fulfilled.**

(Constantin 2006, *Invent. Math.* **166**)
(Constantin & Strauss 2010, *Comm. Pure Appl. Math.* **63**)

Convergence at the Trough

Let be the iterations for $x = \pi/k$ **and** $y \in [-d, -b]$**:**

$$y \;=\; F(y) \;\equiv\; g^{-1}\,\mathrm{Re}\{\mathfrak{P}(\pi/k + \mathrm{i}y)\} \;-\; d$$

Convergence occurs if $-1 < F_y < 1$ **with:**

$$F_y \;=\; 1 \;+\; p_y(\pi/k, y)\,/\,g \;=\; -\,u(\pi/k, y)\,u_y(\pi/k, y)\,/\,g$$

Since $? < p_y \leqslant -g$ **under the trough, the upper inequality is fulfilled, but the lower one is likely.**

Convergence for the Surface

Let be the iterations for $y \in [-d, \eta(x)]$**:**

$$y \;=\; G(y) \;\equiv\; \frac{B}{g} \;-\; \frac{\sqrt{(\,B \,-\, g\,a\,)^2 \,-\, 2\,g\,\operatorname{Im}\{\mathfrak{Q}(x+\mathrm{i}y)\}}}{g}$$

Convergence occurs if $-1 < G_y < 1$ **with:**

$$G_y \;=\; \frac{p \,+\, g\,y \,+\, v^2}{2\,p \,+\, g\,y \,+\, u^2 \,+\, v^2}$$

Condition $G_y > -1$ **yields** $p + v^2 + B > 0$ **: OK.**

Condition $G_y < 1$ **yields** $p + u^2 > 0$ **: OK, except at the crest of the highest wave.**

Limiting Waves

For the surface iterations:

$$G_{yy} \;=\; \frac{g\,G_y \,+\, p_y \,+\, g \,+\, 2\,v\,v_y}{2\,p \,+\, g\,y \,+\, u^2 \,+\, v^2} \;=\; \frac{g\,G_y \,+\, \mathrm{Im}\{w\,w_z\}}{B \,-\, g\,y}$$

Limiting Waves

For the surface iterations:

$$G_{yy} \;=\; \frac{g\,G_y \,+\, p_y \,+\, g \,+\, 2\,v\,v_y}{2\,p \,+\, g\,y \,+\, u^2 \,+\, v^2} \;=\; \frac{g\,G_y \,+\, \mathrm{Im}\{w\,w_z\}}{B \,-\, g\,y}$$

Local expansion at an angular crest (Stokes 1880):

$$\eta \;=\; a \;-\; 3^{-1/2}\,|x| \;+\; O(x^{\nu})\,,$$
$$w^2 \;=\; \mathrm{i}\,g\,(z-\mathrm{i}a) \;+\; O((z-\mathrm{i}a)^{\nu})\,,$$

where $\nu \approx 2.204$ **is the smallest root of (Grant 1973)**

$$\sqrt{3}\,\tan(\nu\pi/3) \;=\; -1 \;-\; 2/\nu$$

Thus $G_{yy} = 3/2a > 0$ **at the crest: Iterations converge from the fluid side** $y < a$ **(and diverge from above** $y > a$**).**

Limiting Waves - Continued

For the crest iterations:

$$F_{yy} \ = \ \frac{p_{yy}}{g} \ = \ \frac{\mathrm{Re}\{(w^2)_{zz}\}}{2\,g}$$

Thence $F_{yy} = 0$ at the crest and F_{yyy} is needed to conclude on the convergence.

Limiting Waves - Continued

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$$F_{yy} \;=\; \frac{p_{yy}}{g} \;=\; \frac{\mathrm{Re}\{(w^2)_{zz}\}}{2\,g}$$

Thence $F_{yy} = 0$ at the crest and F_{yyy} is needed to conclude on the convergence.

But $F_{yyy} \to \infty$ at the crest.

Summary

Problem reformulation.

Analytic resolution.

Easy numerical computations.

Thanks to the w^2 and $\mathfrak{Q}$ functions.

References:

Clamond & Constantin 2013, *J. Fluid Mech.* **714**, pp. 463–475.

Clamond, D. 2013. *J. Fluid Mech.* **726**, pp. 547–558.

Perspectives

Practical applications:

- **Noisy data, outliers.**
- **Unstationary signals.**
- **More complex physics** (e.g., viscosity, breaking).

Generalizations:

- **Rotational flows: Feasible.**
- **Unsteady motions: Perhaps.**
- **Three-dimensional case: More challenging.**

That's it!
Thank you.